Math 5553, Homework 2, Due on 2/20/2014

1. (2 points) Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be two unitary matrices. Show that the $(m+n)\times(m+n)$ matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is also unitary.

2. (4 points) You may use the facts that $det(AB) = det(A)det(B)$ and $det(A^*) = \overline{det(A)}$.

   (a) What are the singular values of a unitary matrix?

   (b) Let $A \in \mathbb{C}^{m\times m}$ be an invertible matrix with singular value decomposition $A = U\Sigma V^*$. What are the singular values of A^{-1}?

   (c) Prove that for any unitary matrix U, we have $|det(U)| = 1$.

   (d) If $A \in \mathbb{C}^{m\times m}$ has singular values $\sigma_1, \sigma_2, \ldots, \sigma_m$, prove that $|\det(A)| = \sigma_1\sigma_2\cdots\sigma_m$. What if A is a real-valued matrix?

3. (4 points) Find an orthogonal projector that maps all 3-dim vectors onto the plane $2x - 4y + 3z = 0$.

4. (4 points) Calculate the computational cost of the Householder QR factorization (Algorithm 10.1).

5. (6 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

   where ε is a positive real number smaller than the machine epsilon. In finite precision arithmetic, we know that 1 plus/minus any positive real number smaller than the machine epsilon will still be equal to 1. We assume all other operations follow the usual arithmetic rules. For example, the Euclidean length of $[\varepsilon, \varepsilon, 0, 0]$ is $\sqrt{2}\varepsilon$. However, to simulate the finite precision arithmetic, the Euclidean length of $[1, \varepsilon, 0, 0]$ should be $\sqrt{1+\varepsilon^2} = \sqrt{1} = 1$.

   Use the classical Gram-Schmidt process and the modified Gram-Schmidt process to compute the reduced QR factorization of matrix A. Do they return the same results? Which one is the correct answer?

   (The following is not part of this assignment. You do not need to do it. Remember that in exact arithmetic, the classical and the modified GS methods are the same. If you are interested, try to figure out why the classical GS method fails for this problem in finite precision arithmetic, while the modified method still works. A good strategy is to first compute the QR factorization under exact arithmetic, and then compare the results.)