Math 5553, Homework 4, Due on April/11/2013

1. (12 points) Consider a 20×20 symmetric positive definite, tridiagonal matrix

$$A = \begin{bmatrix} 21 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 20 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 19 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 18 & -1 & \dots & 0 \\ & & & \ddots & \ddots & \ddots & \\ 0 & \dots & & & -1 & 3 & -1 \\ 0 & \dots & & & 0 & -1 & 2 \end{bmatrix}$$

You can use Matlab command `eig(A)` to check the eigenvalues of A, and consider these as "exact"(high precision approximation) eigenvalues.

   (a) Write Matlab functions for the "pure" QR algorithm, the QR algorithm with Rayleigh quotient shift, and the QR algorithm with Wilkinson shift. Apply these three functions to A. What do they converge to?

   (b) Now let us check the convergence rate of the QR algorithms. For each QR algorithm, locate the eigenvalue that $A^{(k)}(20,20)$ converges to as $k \to \infty$. Denote this eigenvalue as λ. Plot the value of $|\lambda - A^{(k)}(20,20)|$ versus k for $k = 1, \dots, 20$. This gives the history of convergence. Plot the history of convergence of all three QR algorithms in one single graph (use `semilogy` instead of `plot` to better reveal the rate of convergence).

   You should observe that QR algorithm with shift has a faster convergence rate for $A^{(k)}(20,20)$. Do you think this is also true for other $A^{(k)}(i,i)$, $i = 1, \dots, 19$. Note that each $A^{(k)}(i,i)$ shall converge to one eigenvalue of A. Pick a few values of i and plot its history of convergence, using all three QR algorithms, then draw your conclusion.

   (c) Now let us check the convergence rate of $A^{(k)}(20,19)$, the off-diagonal entry on the last row. According to what we have discussed in class, it should converge to 0 as $k \to \infty$. Plot its history of convergence for $k = 1, \dots, 20$, using all three QR algorithms. What do you observe?

   Again, how about the convergence rate of other off-diagonal entries $A^{(k)}(i+1,i)$?

2. (8 points) Given matrix

$$A = \begin{bmatrix} 7 & 3 & 4 & -11 & -9 & -2 \\ -6 & 4 & -5 & 7 & 1 & 12 \\ -1 & -9 & 2 & 2 & 9 & 1 \\ -8 & 0 & -1 & 5 & 0 & 8 \\ -4 & 3 & -5 & 7 & 2 & 10 \\ 6 & 1 & 4 & -11 & -7 & -1 \end{bmatrix}$$

   (a) Write a Matlab program that computes an upper Hessenberg matrix H similar to A.

   (b) Apply the QR algorithms (without shift, with Rayleigh quotient shift, with Wilkinson shift) to H. What do they converge to? What are the eigenvalues of matrix A?