Math 5553, Homework 3, Due on 3/26/2013

1. (4 points) Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & -2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}.$$

   Solve (by hand) the problem $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination with partial pivoting.

2. (4 points) The spectral radius of an $m \times m$ matrix A is defined by

$$\rho(A) = \max_{1 \le i \le m} |\lambda_i|$$

   where λ_i, $i = 1, \ldots, m$ are the eigenvalues of A.

   (a) Prove that for any induced matrix norm $\|\cdot\|$, we have $\rho(A) \le \|A\|$.

   (b) Prove that $\lim_{k\to\infty} \|A^k\| = 0$ if and only if $\rho(A) < 1$. (Hint: use Schur factorization)

3. (4 points) Let a, b, c, d, e, f be real numbers and $a > 0$, $d > 0$, $f > 0$. Define a symmetric matrix A as follows:

$$A = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 + d^2 & bc + de \\ ac & bc + de & c^2 + e^2 + f^2 \end{bmatrix}$$

   Apply Algorithm 23.1 to this matrix, write down each step, and show that it computes the Cholesky factorization of this matrix.

4. (8 points) Suppose a quantity $x(h) \approx ch^r$, where c and r are non-negative constants. In this case we usually say $x = O(h^r)$. For example, when h takes the value of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ and x takes the value 0.803, 0.197, 0.051, 0.0124, we can easily see that r should be around 2. Or in other words, the given set of data satisfies $x(h) = O(h^2)$. The rule is obvious in this case since every time we reduce h by a factor of 2, x will be reduced approximately by a factor of 4.

   However, the value of r is much less obvious if we are given the following set of data:

$$h = \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{20}, \text{ and } x = 0.185,\ 0.0853,\ 0.0486,\ 0.0313$$

   Plotting this set of data using `loglog(h,x)`. The result is almost a straight line, which indicates that the $\ln h$-$\ln x$ relation is linear. Indeed, by taking nature log of $x \approx ch^r$, we have $\ln x \approx r \ln h + \ln c$. Clearly, for any given set of data, we can then use the least squares method (the line fitting) to find r and c. Compute r and c for the above given data.

   Then, test you algorithm on another set of data:

$$h = \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{20}, \text{ and } x = 0.12,\ 0.0672,\ 0.044,\ 0.0316$$