Math 5553, Homework 1, Due on 1/31/2013

1. (4 points) Prove that the $n \times n$ Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

is invertible if and only if x_i, $1 \le i \le n$, are distinct numbers.

2. (6 points) Consider vector norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Prove that for all $\mathbf{x} \in \mathbb{C}^m$,

$$\begin{aligned} \|\mathbf{x}\|_2 &\le \|\mathbf{x}\|_1 \le \sqrt{m}\|\mathbf{x}\|_2 \\ \|\mathbf{x}\|_\infty &\le \|\mathbf{x}\|_2 \le \sqrt{m}\|\mathbf{x}\|_\infty \\ \|\mathbf{x}\|_\infty &\le \|\mathbf{x}\|_1 \le m\|\mathbf{x}\|_\infty \end{aligned}$$

3. (4 points) Let $\|\cdot\|_{(m)}$, $\|\cdot\|_{(n)}$, $\|\cdot\|_{(r)}$ be vector norms on $\mathbb{C}^m$, $\mathbb{C}^n$ and $\mathbb{C}^r$, respectively. Let $\|\cdot\|_{(m,n)}$, $\|\cdot\|_{(n,r)}$, $\|\cdot\|_{(m,r)}$ be induced matrix norms as defined in class. Prove that for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times r}$,

$$\|AB\|_{(m,r)} \le \|A\|_{(m,n)}\|B\|_{(n,r)}.$$

4. (6 points) Consider the following $n \times n$ symmetric tridiagonal matrix:

$$S_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 & 0 \\ & & & \ddots & \ddots & & \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix},$$

It is known that all eigenvalues of S_n are real and positive. For such a matrix, the ratio of the largest eigenvalue to the smallest one (condition number) is an important quantity and can be computed using the Matlab function `cond`. For example, `cond(A)` returns the condition number of matrix A. Type `help cond` under Matlab for more information.

The condition number of a matrix A roughly measures how accurate we can solve the linear system $A\mathbf{x} = \mathbf{b}$. Consider the following experiment. Given matrix A. Define an n-dimentional column vector $\mathbf{x} = [1, 1, \cdots, 1]^t$. Compute $A\mathbf{x}$ to get vector $\mathbf{b}$. Now, use matlab command `A\b` to solve for $A^{-1}\mathbf{b}$. Theoretically, we should get exactly $\mathbf{x}$. However, due to round-off errors in finite-precision computation, the command `A\b` usually only gives an approximate solution, let's call it $\tilde{\mathbf{x}}$, to the exact solution $\mathbf{x}$. It is interesting and also important to investigate the error $\|\tilde{\mathbf{x}} - \mathbf{x}\|_2$, where $\|\cdot\|_2$ is the vector 2-norm. In Matlab, the 2-norm of any given vector $\mathbf{x}$ can be computed using the command `norm(x)`.

   (a) For $n = 5, 6, 7, \cdots, 50$, conduct the above experiment for matrix S_n. You can easily define S_n in Matlab using the command `diag`. Try `help diag` for how to use this command. Compute the error $\|\tilde{\mathbf{x}} - \mathbf{x}\|_2$ for each n and plot them in a graph, with x-axis denoting n and y-axis denoting the error. Also, compute the condition number of each S_n, and plot them versus n, in a separate graph.

(b) Recall the $n \times n$ Hilbert matrix

$$H_n = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

The Hilbert matrix is symmetric, and all of its eigenvalues are real and positive. The only trouble is that, Hilbert matrix is well-known for being "ill-conditioned".

Hilbert matrix can easily be generated in Matlab using the command `hilb(n)`. Now, repeat the experiment in part (a), (b) for the Hilbert matrix, but with $n = 2, 3, 4, \cdots, 12$. Report the error and condition numbers as two separate graphs.

(c) Compare the graphs from part (a) and (b). What's your conclusion?