Math 5553, Homework 5, Due on 4/19/2012

1. (8 points) Let U and V be two $m \times m$ unitary matrices. Find the inverse of

$$\begin{bmatrix} V & V \\ U & -U \end{bmatrix}$$

2. (12 points) From the previous homework assignment, we know that the $m \times m$ matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

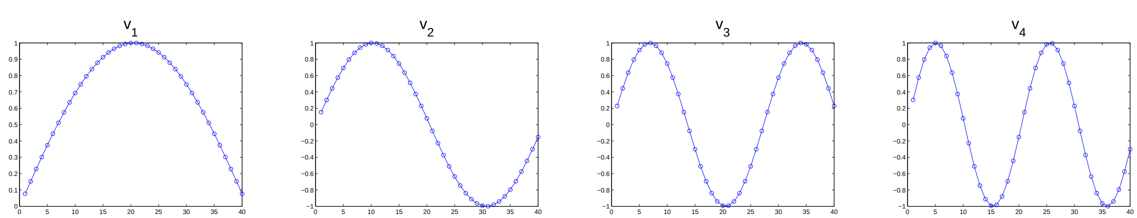
has eigen values $\lambda_k = 2 - 2\cos\frac{k\pi}{m+1}$, for $k = 1, 2, \ldots, m$, and the corresponding eigenvectors

$$\mathbf{v}_k = \begin{bmatrix} \sin\frac{k\pi}{m+1} \\ \sin\frac{2k\pi}{m+1} \\ \sin\frac{3k\pi}{m+1} \\ \vdots \\ \sin\frac{mk\pi}{m+1} \end{bmatrix}$$

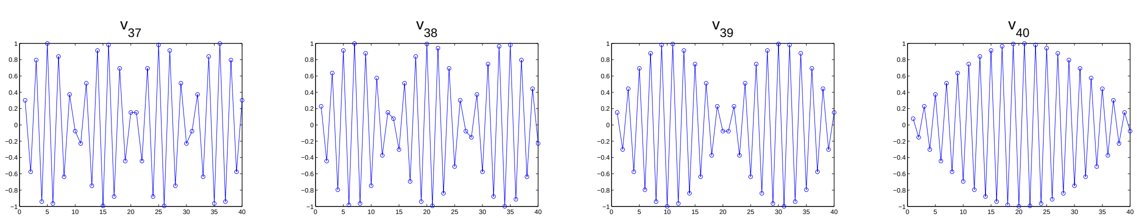
It is also clear that

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_m < 4$$

Using Matlab to draw each eigenvector, we have the following results. Here we picked $m = 40$ and drawn the first 4 and the last 4 eigenvectors.



$\cdots \cdots \cdots \cdots$



It is quite obvious from the above graphs, that eigenvectors corresponding to small eigenvalues tend to be "smooth" while eigenvectors corresponding to large eigenvalues tend to be "highly oscillating". Many matrices derived from physics backgrounds demonstrate similar properties. (However, do keep in mind that not all matrices have such a property.) If a matrix does demonstrate this property, then its eigenvalues can be viewed as the "frequency" of the corresponding eigenvectors. High frequency eigenvectors are highly oscillating.

It is known that stationary iterative solvers such as Jacobi, Gauss-Seidel, and SOR methods reduces the "high frequency" part (the part corresponding to large eigenvalues) of the error $\mathbf{e}_i = \mathbf{x} - \mathbf{x}_i$ faster than the "low frequency" part (the part corresponding to small eigenvalues). Such a "smoothing effect" makes these iterative solvers extremely useful in the design of certain preconditioners, for example, the multigrid preconditioner. To demonstrate the "smoothing effect" of the Gauss-Seidel method, we design the following experiments. In all the following experiments, set the initial guess $\mathbf{x}_0 = \mathbf{0}$.

(a) Let $m = 40$ and consider the linear system $A\mathbf{x} = \mathbf{b}$. First, we set the exact solution $\mathbf{x} = \mathbf{v}_1$ and hence $\mathbf{b} = A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. Apply the Gauss-Seidel iterative method for this problem. How many iteration steps is required to reach a relative residual of 10^{-3}? Then, repeat the experiment using $\mathbf{x} = \mathbf{v}_m$ and the relative residual is still 10^{-3}. This time, how many iteration steps is required?

(b) Let $m = 40$. Now let us check what happens if the exact solution is set to be $\mathbf{x} = \mathbf{v}_1 + \mathbf{v}_m$. This solution contains a "low frequency" part and a "high frequency" part. Apply the Gauss-Seidel iteration and only iterate for 6 steps. After each iteration step, plot the error $\mathbf{e}_i = \mathbf{x} - \mathbf{x}_i$. Compare the graphs and explain what you have observed.