Math 5553, Homework 4, Due on 4/3/2012

1. (6 points) Consider the $m \times m$ symmetric tridiagonal matrix

$$T = \begin{bmatrix} a & b & 0 & 0 & \cdots & 0 \\ b & a & b & 0 & \cdots & 0 \\ 0 & b & a & b & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & a \end{bmatrix}$$

where a and b are real numbers.

Prove that its eigenvalues takes the form $\lambda_k = a + 2b\cos\frac{k\pi}{m+1}$, for $k = 1, 2, \ldots, m$, and the corresponding eigenvector is

$$\mathbf{x}_k = \begin{bmatrix} \sin\frac{k\pi}{m+1} \\ \sin\frac{2k\pi}{m+1} \\ \sin\frac{3k\pi}{m+1} \\ \vdots \\ \sin\frac{mk\pi}{m+1} \end{bmatrix}$$

Hint: Since the form of eigenvalues and eigenvectors are given, you only need to verify that $T\mathbf{x}_k = \lambda_k \mathbf{x}_k$ for all $k = 1, 2, \ldots, m$. You probably need to use some trigonometric identities, which can be found on

`http://en.wikipedia.org/wiki/List_of_trigonometric_identities`

2. (6 points) According to Problem 1, clearly for $a = 2$ and $b = -1$, matrix T is symmetric positive definite. We know that the Taylor expansion of cosine is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Use this and the eigenvalues from Problem 1 to prove that $cond(T) = O(m^2)$. That is, when m gets larger, $cond(T)$ grows in a rate proportional to m^2.

3. (8 points) Let $a = 2$, $b = -1$, and $m = 20$. Write a Matlab program to solve $T\mathbf{x} = \mathbf{f}$ using the Cholesky factorization with backward and forward substitutions. Here $\mathbf{f} = [1, 1, \cdots, 1]^t$.

Remark: The tridiagonal matrix T has a "band" structure. Indeed, for such band matrices, LU factorization (include Cholesky factorization) reserves the band structure. Namely, matrices L and U (or R in the Cholesky factorization) also have the band structure. This is often used to reduce the storage and computational costs. Although it is not mandatory for this homework assignment, you are encouraged to incorporate this feature in your program. (Sorry, no extra credit for doing so.)