Math 5553, Homework 1, Due on 1/31/2012

1. (4 points) Prove the triangle inequality holds for the vector p-norm $\|\cdot\|_p$, for $p = 1$ and ∞.

2. (6 points) Consider vector norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. Prove that for all $\mathbf{x} \in \mathbb{C}^m$,

$$\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le \sqrt{m}\|\mathbf{x}\|_2$$
$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \sqrt{m}\|\mathbf{x}\|_\infty$$
$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le m\|\mathbf{x}\|_\infty$$

3. (10 points) Consider the following two type of $n \times n$ tridiagonal matrices:

$$M_n = \begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 0 & \cdots & 0 & 0 \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix}, \quad S_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 & 0 \\ & & & \ddots & \ddots & & \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix},$$

Notice that both of them are symmetric. Furthermore, we also know that all their eigenvalues are real and positive. For such matrices, the ratio of the largest eigenvalue to the smallest one (condition number) is an important quantity and can be computed using the Matlab function `cond`. For example, `cond(A)` returns the condition number of matrix A.

The condition number of a matrix A roughly measures how accurate we can solve the linear system $A\mathbf{x} = \mathbf{b}$. Consider the following experiment. Define an n-dimentional column vector $\mathbf{x} = [1, 1, \cdots, 1]^t$. Compute $A\mathbf{x}$ to get vector $\mathbf{b}$. Now, use matlab command `A\b` to solve for $A^{-1}\mathbf{b}$. Theoretically, we should get exactly $\mathbf{x}$. However, due to round-off errors in finite-precision computation, we usually only get an approximate solution, let's call it $\tilde{\mathbf{x}}$, to the exact solution $\mathbf{x}$. It is interesting and also important to investigate the error $\|\tilde{\mathbf{x}} - \mathbf{x}\|_2$, where $\|\cdot\|_2$ is the vector 2-norm (Euclidean length). In Matlab, the 2-norm of any given vector $\mathbf{x}$ can be computed using the command `norm(x)`.

   (a) For $n = 5, 6, 7, \cdots, 50$, conduct the above experiment for matrix M_n. You can easily define M_n in Matlab using the command `diag`. Try `help diag` for how to use this command. Compute the error $\|\tilde{\mathbf{x}} - \mathbf{x}\|_2$ for each n and plot them in a graph, with x-axis denoting n and y-axis denoting the error. Also, compute the condition number of each M_n, and plot them versus n, in a separate graph.

   (b) Repeat (a) for S_n.

   (c) Recall the $n \times n$ Hilbert matrix

$$H_n = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

The Hilbert matrix is symmetric, and all of its eigenvalues are real and positive. The only trouble is that, Hilbert matrix is well-known for being "ill-conditioned".

Hilbert matrix can easily be generated in Matlab using the command `hilb(n)`. Now, repeat the experiment in part (a), (b) for the Hilbert matrix, but with $n = 2, 3, 4, \cdots, 12$. Report the error and condition numbers as two separate graphs.

(d) Compare the six graphs from part (a), (b) and (c). What's your conclusion?