Homework 5

1. Examine the stability of the Douglass-Rachford method (7.3.10).

2. (Problem 3, Numerical Analysis Comprehensive Exam, August, 2007) Let function \( u(x, y) \) be the solution to the following Dirichlet problem (which means Poisson equation with Dirichlet boundary conditions):

\[
\begin{align*}
\nabla^2 u &\triangleq u_{xx} + u_{yy} = f & \text{in } \Omega, \\
u &\equiv 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega = \{(x, y) \mid |x| + |y| < 1\} \). Let and grid function \( v \) be defined on the uniform grid over \( \bar{\Omega} \triangleq \{(x, y) \mid |x| + |y| \leq 1\} \) (the closure of \( \Omega \)). In other words, \( v_{lm} \) denotes the value of \( v \) at a point \( (x_l, y_m) = (hl, hm) \) in \( \bar{\Omega} \), where \( h \) is the step size. Denote \( \nabla^2_h \) to be the standard five-point Laplacian. Suppose that the grid function \( v \) is the solution to \( \nabla^2_h v = f \) over the interior of \( \Omega \), and \( v_{lm} = 0 \) on the boundary of \( \Omega \). Prove that there exists a constant \( C_0 > 0 \) independent of \( h \) and \( u \) such that

\[
\| u - v \|_\infty \leq C_0 h^2 \| \partial^4 u \|_\infty.
\]

Here the maximum norm \( \| \cdot \|_\infty \) is taken on all grid points in \( \Omega \).

3. Solve the Poisson’s equation in \( \Omega = (0, 1) \times (0, 1) \):

\[
\begin{align*}
-\Delta u &= 2\pi^2 \sin(\pi x) \sin(\pi y) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

by the standard five-point finite difference scheme on a uniform \( M \times M \) grid. The step size is \( h = 1/M \). The exact solution for this problem is \( u = \sin(\pi x) \sin(\pi y) \).

(a) Compute the maximum norm of error \( \| u - v \|_\infty \) on grid points, where \( v \) is the finite difference solution. Report the error for \( 16 \times 16, 32 \times 32, 64 \times 64 \) and \( 128 \times 128 \) grids. Does your result agree with the error estimate

\[
\| u - v \|_\infty = O(h^2).
\]

(b) Plot the numerical solution on a \( 20 \times 20 \) grid.

Note:

- You may use the Matlab build-in function “delsq” to generate the stiffness matrix. However, you will need to read the help file and make sure you use it correctly.
- To solve a linear system \( Ax = f \), where \( x \), and \( f \) are \( n \)-dim column vectors, you can use the Matlab command “\( x = A \backslash f \)”.
- A useful Matlab command is “reshape”, which returns a matrix whose elements are taken column-wise from a given vector or matrix. For example, let column vector \( x = [1, 2, 3, 4]^t \). “\( y=\text{reshape}(x,2,2) \)” will return a \( 2 \times 2 \) matrix \( y = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \). Then you can use command “surf(y)” to plot the matrix as a surface.