Homework 5

1. Examine the stability of the Douglass-Rachford method (7.3.10).

2. (Problem 3, Numerical Analysis Comprehensive Exam, August, 2007) Let function $u(x, y)$ be the solution to the following Dirichlet problem (which means Poisson equation with Dirichlet boundary conditions):
$$\begin{cases} \nabla^2 u \triangleq u_{xx} + u_{yy} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
where $\Omega = \{(x, y) \,|\, |x| + |y| < 1\}$. Let and grid function v be defined on the uniform grid over $\bar{\Omega} \triangleq \{(x, y) \,|\, |x| + |y| \leq 1\}$ (the closure of Ω). In other words, v_{lm} denotes the value of v at a point $(x_l, y_m) = (hl, hm)$ in $\bar{\Omega}$, where h is the step size. Denote ∇_h^2 to be the standard five-point Laplacian. Suppose that the grid function v is the solution to $\nabla_h^2 v = f$ over the interior of Ω, and $v_{lm} = 0$ on the boundary of Ω. Prove that there exists a constant $C_0 > 0$ independent of h and u such that
$$\|u - v\|_\infty \leq C_0 h^2 \|\partial^4 u\|_\infty.$$
Here the maximum norm $\|\cdot\|_\infty$ is taken on all grid points in Ω.

3. Solve the Poisson's equation in $\Omega = (0, 1) \times (0, 1)$:
$$\begin{cases} -\Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
by the standard five-point finite difference scheme on a uniform $M \times M$ grid. The step size is $h = 1/M$. The exact solution for this problem is $u = \sin(\pi x) \sin(\pi y)$.

   (a) Compute the maximum norm of error $\|u - v\|_\infty$ on grid points, where v is the finite difference solution. Report the error for 16×16, 32×32, 64×64 and 128×128 grids. Does your result agree with the error estimate
$$\|u - v\|_\infty = O(h^2).$$

   (b) Plot the numerical solution on a 20×20 grid.

   Note:

   - You may use the Matlab build-in function "delsq" to generate the stiffness matrix. However, you will need to read the help file and make sure you use it correctly.
   - To solve a linear system $Ax = f$, where x, and f are n-dim column vectors, you can use the Matlab command "x = A\ f".
   - A useful Matlab command is "reshape", which returns a matrix whose elements are taken column-wise from a given vector or matrix. For example, let column vector $x = [1, 2, 3, 4]^t$. "y=reshape(x,2,2)" will return a 2×2 matrix $y = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then you can use command "surf(y)" to plot the matrix as a surface.