1 (a) This part is trivial and hence skipped.

(b) There is no contradition. The definition of stability states that the norm should be bounded for any choices of initial conditions. For multi-step schemes such as the leap-frog scheme, V_m^1 should also be viewed as initial conditions.

2. By the Von Neumann analysis,

$$g^2 = e^{-2i\theta} - (a\lambda-1)(1-e^{-2i\theta})g$$

$$\Rightarrow \quad g^2 + (a\lambda-1)(e^{i\theta}-e^{-i\theta})e^{-i\theta}g - e^{-2i\theta} = 0$$

$$\Rightarrow \quad (e^{i\theta}g)^2 + 2(a\lambda-1)\,i\sin\theta\,(e^{i\theta}g) - 1 = 0$$

$$\Rightarrow \quad e^{i\theta}g = -(a\lambda-1)\,i\sin\theta \pm \sqrt{1-(a\lambda-1)^2\sin^2\theta}$$

① If $|a\lambda-1|<1$, then $g_1 \neq g_2$ for all θ and

$$|g_1| = |g_2| = 1$$

The scheme is stable

② If $|a\lambda-1|>1$, then $|g_1(\frac{\pi}{2})| = |g_2(\frac{\pi}{2})| > 1$

The scheme is unstable

③ If $|a\lambda-1|=1$, then

$$\cancel{|g_1(\tfrac{\pi}{2})| =}\ g_1(\tfrac{\pi}{2}) = g_2(\tfrac{\pi}{2})$$

and $|g_1(\frac{\pi}{2})| = |g_2(\frac{\pi}{2})| = 1$

The scheme is unstable.

Combining all the above, the scheme is stable iff $|a\lambda-1|<1$

3. By the von Neumann analysis

$$g = 1 - ak\left(1 - \frac{h^2}{6}\left(-\frac{4}{h^2}\sin^2\frac{\theta}{2}\right)\right)\frac{i}{h}\sin\theta$$
$$+ \frac{a^2k^2}{2}\left(\left(\frac{4}{3} + a^2\lambda^2\right)\left(-\frac{4}{h^2}\sin^2\frac{\theta}{2}\right) - \left(\frac{1}{3} + a^2\lambda^2\right)\left(\frac{i}{h}\sin\theta\right)^2\right)$$
$$= 1 - 2a^2\lambda^2\left(\frac{4}{3} + a^2\lambda^2\right)\sin^2\frac{\theta}{2} + \frac{a^2\lambda^2}{2}\left(\frac{1}{3} + a^2\lambda^2\right)\sin^2\theta$$
$$- i\,a\lambda\left(1 + \frac{2}{3}\sin^2\frac{\theta}{2}\right)\sin\theta$$

Therefore

$$\mathrm{Re}\,g = 1 - 2a^2\lambda^2\left(\frac{4}{3} + a^2\lambda^2\right)\sin^2\frac{\theta}{2} + 2a^2\lambda^2\left(\frac{1}{3} + a^2\lambda^2\right)\sin^2\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right)$$
$$= 1 - 2a^2\lambda^2\sin^2\frac{\theta}{2} - 2a^2\lambda^2\left(\frac{1}{3} + a^2\lambda^2\right)\sin^4\frac{\theta}{2}$$
$$= 1 - C_1\sin^2\frac{\theta}{2} - C_2\sin^4\frac{\theta}{2}$$

where $C_1 = 2a^2\lambda^2$, $\quad C_2 = 2a^2\lambda^2\left(\frac{1}{3} + a^2\lambda^2\right)$

$$|\mathrm{Im}\,g|^2 = a^2\lambda^2\left(1 + \frac{2}{3}\sin^2\frac{\theta}{2}\right)^2\sin^2\theta$$
$$= 4a^2\lambda^2\left(1 + \frac{4}{3}\sin^2\frac{\theta}{2} + \frac{4}{9}\sin^4\frac{\theta}{2}\right)\left(\sin^2\frac{\theta}{2} - \sin^4\frac{\theta}{2}\right)$$
$$= 4a^2\lambda^2\sin^2\frac{\theta}{2} + \frac{4}{3}a^2\lambda^2\sin^4\frac{\theta}{2}$$
$$- \frac{32}{9}a^2\lambda^2\sin^6\frac{\theta}{2} - \frac{16}{9}a^2\lambda^2\sin^8\frac{\theta}{2}$$

Hence

$$|g|^2 = |\mathrm{Re}\,g|^2 + |\mathrm{Im}\,g|^2$$
$$= 1 \cancel{- 2C_1\sin^2\frac{\theta}{2}} + \cancel{\left(C_1^2\sin^4\frac{\theta}{2} - 2C_2\sin^4\frac{\theta}{2}\right)} + 2C_1C_2\sin^6\frac{\theta}{2} + C_2^2\sin^8\frac{\theta}{2}$$
$$\cancel{+ 4a^2\lambda^2\sin^2\frac{\theta}{2}} \cancel{+ \frac{4}{3}a^2\lambda^2\sin^4\frac{\theta}{2}} - \frac{32}{9}a^2\lambda^2\sin^6\frac{\theta}{2} - \frac{16}{9}a^2\lambda^2\sin^8\frac{\theta}{2}$$
$$= 1 - d_1\sin^6\frac{\theta}{2} - d_2\sin^8\frac{\theta}{2}$$

where

$$d_1 = \frac{32}{9}a^2\lambda^2 - 2C_1C_2 = a^2\lambda^2\left(\frac{32}{9} - \frac{8}{3}a^2\lambda^2 - 8a^4\lambda^4\right)$$
$$d_2 = \frac{16}{9}a^2\lambda^2 - C_2^2 = 4a^2\lambda^2\left(\frac{4}{9} - a^2\lambda^2\left(\frac{1}{3} + a^2\lambda^2\right)^2\right)$$

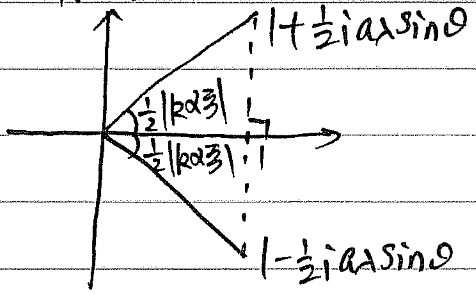
Clearly $d_1 > 0$ iff $|a\lambda| < \left(\frac{\sqrt{17} - 1}{6}\right)^{1/2}$

It is also not hard to see that $d_2 > 0$ for $0 < a^2\lambda^2 < \frac{\sqrt{17} - 1}{6}$

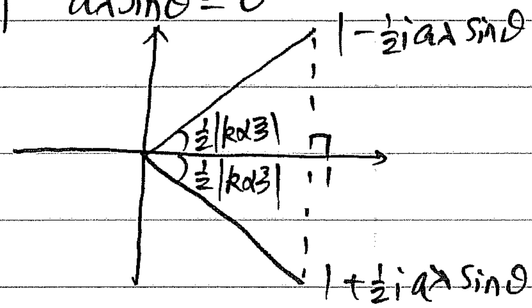
4 It is not hard to see that

$$g = \frac{1 - \frac{1}{2} i a\lambda \sin\theta}{1 + \frac{1}{2} i a\lambda \sin\theta}$$

If $a\lambda \sin\theta > 0$



If $a\lambda \sin\theta \le 0$



In either case, it is not hard to see that

$$\tan\left(\tfrac{1}{2} k\alpha\xi\right) = \tfrac{1}{2} a\lambda \sin\theta$$

The Taylor expansion part is trivial and hence skipped.

5. Notice that

$$|v_m^n - u(t_n, mh)| = \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} g^n \hat{u}_0 \, d\xi - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{imh\xi} e^{q(\xi)t_n} \hat{u}_0 \, d\xi \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \left| \int_{-\pi/h}^{\pi/h} e^{imh\xi} (g^n - e^{q(\xi)t_n}) \hat{u}_0 \, d\xi \right| + \frac{1}{\sqrt{2\pi}} \left| \int_{|\xi| > \pi/h} e^{imh\xi} e^{q(\xi)t_n} \hat{u}_0 \, d\xi \right|$$

$$= I_1 + I_2$$

Clearly,

$$\begin{aligned}
I_1 &\leq \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} |g^n - e^{q(\xi)t_n}| \cdot |\hat{u}_0| \, d\xi \\
&\leq C(t_n)\, n \int_{-\pi/h}^{\pi/h} |g - e^{q(\xi)k}| \cdot |\hat{u}_0| \, d\xi \\
&\leq C(t_n)\, n \int_{-\pi/h}^{\pi/h} k h^r (1+|\xi|)^\rho |\hat{u}_0| \, d\xi \\
&= C(t_n) h^r \int_{-\pi/h}^{\pi/h} (1+|\xi|)^\rho |\hat{u}_0| \, d\xi \\
&\leq C(t_n) h^r \int_{-\infty}^{\infty} (1+|\xi|)^\rho |\hat{u}_0| \, d\xi
\end{aligned}$$

and

$$\begin{aligned}
I_2 &\leq \frac{1}{\sqrt{2\pi}} \int_{|\xi| > \pi/h} |e^{q(\xi)t_n}| \, |\hat{u}_0| \, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{|\xi| > \pi/h} |e^{-ia\xi t_n}| \, |\hat{u}_0| \, d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{|\xi| > \pi/h} |\hat{u}_0| \, d\xi \\
&\leq \frac{1}{\sqrt{2\pi}} (h/\pi)^\rho \int_{|\xi| > \pi/h} |\xi|^\rho |\hat{u}_0| \, d\xi \\
&\leq C h^\rho \int_{-\infty}^{\infty} |\xi|^\rho |\hat{u}_0| \, d\xi
\end{aligned}$$

Combine the estimates for I_1 and I_2, and we have

$$|v_m^n - u(t_n, mh)| \leq C(t_n) h^r \int_{-\infty}^{\infty} (1+|\xi|)^\rho |\hat{u}_0| \, d\xi$$

as long as $\rho \geq r$, which is usually true.

6. The proof is trivial and hence omitted.