Homework 4 Due 12/3/2015

1. (6 points) Given $\xi^* \in (-\frac{\pi}{h}, \frac{\pi}{h})$ and infinite dimensional vectors $\{v_m\}_{m=-\infty}^{\infty}$ and $\{u_m\}_{m=-\infty}^{\infty}$.
   (a) Use the definition of discrete fourier transform from von Neumann analysis to show that $\hat{v}(\xi) = \sqrt{2\pi}\,\delta_{\xi^*}(\xi)$ implies that $v_m = e^{imh\xi^*}$.
   (b) Denote $w_m = e^{imh\xi^*} u_m$ for all m. Prove that $\hat{w}(\xi) = \hat{u}(\xi - \xi^*)$.

2. (6 points) Show that the Crank-Nicolson scheme for $u_t = bu_{xx}$, where $b > 0$, is unconditionally stable. For the refinement path $\mu = const$, find the order of dissipation of the scheme.

3. (8 points) (Problem 6.3.6, part (c)) Consider the following scheme for the parabolic equation $u_t = bu_{xx}$ with $b > 0$,

$$\tilde{v}_m^{n+1/2} = v_m^n + \frac{1}{2} kb\,\delta^2 v_m^n$$

$$v_m^{n+1} = v_m^n + kb\,\delta^2 \tilde{v}_m^{n+1/2}$$

   Show that this scheme is stable if $b\mu \leq 1/2$ and show that $1/2 \leq g \leq 1$.

   (Hint: Recall that g is defined by $\hat{v}^{n+1} = g\hat{v}^n$. You can either (1) express v_m^{n+1} in terms of v_m^n and then compute g; or (2) compute $\hat{v}^{n+1} = G_1(\hat{v}^n, \hat{\tilde{v}}^{n+1/2})$ and $\hat{\tilde{v}}^{n+1/2} = G_2(\hat{v}^n)$, and then compute g from the composite function.)