#1. (a) By definition

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}\, \hat{v}(\xi)\, d\xi$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi}\, \cancel{\sqrt{2\pi}}\, \delta_{\xi^*}(\xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} e^{imh\xi}\, \delta_{\xi^*}(\xi)\, d\xi \qquad \left(\begin{array}{l}\text{Since } \delta_{\xi^*}(\xi) = 0 \\ \text{when } \xi \neq \xi^*\end{array}\right)$$

$$= e^{imh\xi_*} \qquad (\text{property of } \delta\text{-function})$$

(b) By definition

$$\hat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi}\, w_m \cdot h$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi} \left(e^{imh\xi^*} u_m\right) \cdot h$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh(\xi-\xi^*)}\, u_m \cdot h$$

$$= \hat{u}(\xi - \xi^*)$$

#2 The C-N scheme for $u_t = b u_{xx}$ is

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{1}{2} b \left(\delta^2 U_m^{n+1} + \delta^2 U_m^n \right)$$

$$\Rightarrow \quad g - 1 = \frac{1}{2} b\mu \left(-4g \sin^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} \right)$$

$$\Rightarrow \quad g = \frac{1 - 2b\mu \sin^2\frac{\theta}{2}}{1 + 2b\mu \sin^2\frac{\theta}{2}}$$

Since $2b\mu \sin^2\frac{\theta}{2} \geq 0$, it is clear that $|g| \leq 1$. for all values of b, μ, θ. Therefore the scheme is unconditionally stable.

Now, note that

Case I. when $2b\mu \sin^2\frac{\theta}{2} \leq 1$, we have $g \geq 0$ and

$$|g| = g = 1 - \frac{4b\mu}{1 + 2b\mu \sin^2\frac{\theta}{2}} \sin^2\frac{\theta}{2} \leq 1 - \frac{4b\mu}{1+2b\mu} \sin^2\frac{\theta}{2}$$

Case 2, when $2b\mu \sin^2\frac{\theta}{2} > 1$, we have $g < 0$ and

$$|g| = -g = \frac{2b\mu \sin^2\frac{\theta}{2} - 1}{2b\mu \sin^2\frac{\theta}{2} + 1} = 1 - \frac{2}{1 + 2b\mu \sin^2\frac{\theta}{2}}$$

$$\leq 1 - \frac{2}{1+2b\mu} \leq 1 - \frac{2}{1+2b\mu} \sin^2\frac{\theta}{2}$$

Combining Case I & 2, one gets

$$|g| \leq 1 - \min\left\{ \frac{4b\mu}{1+2b\mu}, \frac{2}{1+2b\mu} \right\} \sin^2\frac{\theta}{2} \Rightarrow \text{dissipative of order 2}$$

#3. By von Neumann analysis

$$\widehat{v}^{n+1/2} = \left(1 - 2b\mu \sin^2\frac{\theta}{2}\right)\widehat{v}^{n}$$

$$\widehat{v}^{n+1} = \widehat{v}^{n} - 4b\mu\sin^2\frac{\theta}{2}\,\widehat{v}^{n+1/2}$$

$$\Rightarrow \widehat{v}^{n+1} = \widehat{v}^{n} - 4b\mu\sin^2\frac{\theta}{2}\left(1-2b\mu\sin^2\frac{\theta}{2}\right)\widehat{v}^{n}$$

$$= \left(1 - 4b\mu\sin^2\frac{\theta}{2} + 8b^2\mu^2\sin^4\frac{\theta}{2}\right)\widehat{v}^{n}$$

$$\Rightarrow g = 1 - 4\gamma + 8\gamma^2 \qquad \left(\text{where } \gamma = b\mu\sin^2\frac{\theta}{2}\right)$$

$$= \left(8\gamma^2 - 4\gamma + \frac{1}{2}\right) + \frac{1}{2}$$

$$= \left(2\sqrt{2}\gamma - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}$$

Clearly, $b\mu \le \frac{1}{2} \Rightarrow 0 \le \gamma \le \frac{1}{2}$ (Note we always have $\gamma \ge 0$)

$$\Rightarrow \left(2\sqrt{2}\gamma - \frac{1}{\sqrt{2}}\right)^2 \le \frac{1}{2}$$

$\Rightarrow g \le 1 \Rightarrow$ the scheme is stable

Moreover, it is obvious that $g = \left(2\sqrt{2}\gamma - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} \ge \frac{1}{2}$

$\Rightarrow \frac{1}{2} \le g \le 1$ when $b\mu \le \frac{1}{2}$