Homework 2 Due 11/3/2015

1. (5 points) Show that $\delta_+\delta_- = \delta_-\delta_+ = \delta^2$. Is $(\delta_0)^2 = \delta^2$? What is the order of the approximation error for $(\delta_0)^2 v - v_{xx}$?

2. (8 points) (Problem 4.4.4 in textbook) Show the modified leapfrog scheme
$$\frac{U_m^{n+1} - U_m^{n-1}}{2k} + a\delta_0\left(\frac{U_m^{n+1} + 4U_m^n + U_m^{n-1}}{6}\right) = f_m^n$$
for the one way wave equation is stable if and only if $|a\lambda| < \sqrt{3}$.

   (Hint: You may introduce new variables, for example $b = \frac{a\lambda\sin\theta}{3}$, to simplify the notation during the calculation.)

   (Remark: To solve the problem, you need to show that (1) when $|a\lambda| < \sqrt{3}$, the scheme is stable; (2) when $|a\lambda| = \sqrt{3}$, the scheme is unstable; (3) when $|a\lambda| > \sqrt{3}$, the scheme is unstable. You will get full credit (8 points) as long as you correctly proved two of (1)-(3). You are encouraged to solve all of (1)-(3), though there is no extra credit for doing so.)

3. (7 points) The leapfrog scheme
$$\frac{U_m^{n+1} - U_m^{n-1}}{2k} + a\frac{U_{m+1}^n - U_{m-1}^n}{2h} = 0$$
is known to have a parasitic mode. To control the effect of the parasitic mode, one way is to use the Robert-Asselin filter. The idea is to introduce another grid-function W_m^n and the new algorithm is

   - Set the initial value $U_m^0 = u_0(x_m)$ and compute U_m^1 using a one-step scheme;
   - Set $W_m^0 = U_m^0$
   - For $n = 1, 2, \ldots$, do
   $$\frac{U_m^{n+1} - W_m^{n-1}}{2k} + a\frac{U_{m+1}^n - U_{m-1}^n}{2h} = 0,$$
   $$W_m^n = U_m^n + \frac{\nu}{2}\left(U_m^{n+1} - 2U_m^n + W_m^{n-1}\right),$$
   where $\nu \in [0, 1]$ is a parameter controlling the strength of the filter (0 - no filter; 1 - filter fully open).

   Use the filtered leapfrog scheme to solve
$$\begin{cases} u_t + u_x = 0 & \text{for } x \in [-1, 3],\ t \in [0, 2.4] \\ u(0, x) = \begin{cases} \cos^2(\pi x) & \text{if } |x| \le 0.5 \\ 0 & \text{otherwise} \end{cases} \\ u(t, -1) = 0 & \text{for } t \in [0, 2.4] \end{cases}$$
with $h = 1/20$ and $\lambda = 0.8$. Use forward-time central-space scheme to compute U_m^1. For the right boundary, you can use $U_M^{n+1} = U_{M-1}^{n+1}$. Draw the solution W at the second last time step, i.e., $N - 1$, for $\nu = 0$, 0.1, 0.2, in one graph so that you can compare them. Describe the pros and cons of using the Robert-Asselin filter.

   Remark: An improvement to the Robert-Asselin filter was given by Paul D. Williams in *A proposed modification to the Robert-Asselin time filter. Mon. Wea. Rev., 137(8):2538-2546, 2009.* This improvement resolves the draw-back you observed in the Robert-Asselin filter.