Homework 2 Due 10/6/2015

Remark: if not specified, the coefficient a can be either positive or negative.

1. (5 points) Examine the stability in maximum norm for the Lax-Friedrichs scheme for $u_t + au_x = 0$.

2. (5 points) Determine the order of accuracy of the following scheme

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+2}^n - 3v_{m+1}^n + 3v_m^n - v_{m-1}^n}{h^3} = 0$$

   for the problem $u_t + au_{xxx} = 0$.

3. (5 points) (NA comprehensive exam, Summer 2012) Consider the scheme

$$\frac{1}{2k}\left[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)\right] + \frac{a}{2h}\left[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)\right] = 0$$

   for the one-way wave equation $u_t + au_x = 0$. Here a is a real constant, v_m^n is the value of the grid function defined on $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Prove the scheme is consistent and is stable in 2-norm for all values of $\lambda = k/h$.

4. (5 points) (The weighted average method) One may define new schemes using various combinations of old schemes. The weighted average method is one of them. Consider the backward-time central-space scheme and the forward-time central-space scheme for $u_t + au_x = 0$:

$$\text{BT-CS:} \qquad \frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2h} = 0,$$

$$\text{FT-CS:} \qquad \frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^n - U_{m-1}^n}{2h} = 0.$$

   Let $0 \le s \le 1$ be a parameter. Multiply the BT-CS scheme by s and the FT-CS by $1 - s$ and then add them together. This gives a weighted average of these two schemes. Note that when $s = 0$, the scheme is just FT-CS, when $s = 1$, the scheme is just BT-CS. By setting $s = 1/2$, one gets exactly the Crank-Nicolson scheme.

   Analyze the stability of this weighted average scheme for $0 \le s \le 1$.