#1. The Lax-Friedrichs scheme for $u_t + a u_x = 0$ is

$$\frac{U_m^{n+1} - \frac{1}{2}\left(U_{m+1}^n + U_{m-1}^n\right)}{k} + a\,\frac{U_{m+1}^n - U_{m-1}^n}{2h} = 0$$

$$\Rightarrow \quad U_m^{n+1} = \frac{1}{2}(1+a\lambda)\,U_{m-1}^n + \frac{1}{2}(1-a\lambda)\,U_{m+1}^n$$

Assume $\max\limits_m |U_m^{n+1}|$ is reached at m^*. i.e.

$$\left|U_{m^*}^{n+1}\right| = \max_m \left|U_m^{n+1}\right|$$

then

$$\|\vec{U}^{n+1}\|_\infty = \left|U_{m^*}^{n+1}\right| = \left|\frac{1}{2}(1+a\lambda)\,U_{m^*-1}^n + \frac{1}{2}(1-a\lambda)\,U_{m^*+1}^n\right|$$

$$\leq \frac{1}{2}|1+a\lambda|\,\left|U_{m^*-1}^n\right| + \frac{1}{2}|1-a\lambda|\,\left|U_{m^*+1}^n\right|$$

$$\leq \frac{1}{2}|1+a\lambda|\,\max_m\left|U_m^n\right| + \frac{1}{2}|1-a\lambda|\,\max_m\left|U_m^n\right|$$

$$= \frac{1}{2}\Big(|1+a\lambda| + |1-a\lambda|\Big)\,\|\vec{U}^n\|_\infty$$

Case 1. If $|a\lambda| \leq 1$, then both $1+a\lambda$ and $1-a\lambda$ are non-negative, and

$$\|\vec{U}^{n+1}\|_\infty \leq \frac{1}{2}\big((1+a\lambda)+(1-a\lambda)\big)\,\|\vec{U}^n\|_\infty = \|\vec{U}^n\|_\infty$$

$\Rightarrow$ ~~the~~ stable

Case 2. if $a\lambda > 1$, then

$$\|\vec{U}^{n+1}\|_\infty \leq \frac{1}{2}\big((1+a\lambda)+(a\lambda-1)\big)\,\|\vec{U}^n\|_\infty = a\lambda\,\|\vec{U}^n\|_\infty = \frac{a}{h}k\,\|\vec{U}^n\|_\infty$$

it can not be bounded by $(1+Kk)\|\vec{U}^n\|_\infty$ $\Rightarrow$ unstable

Case 3 If $a\lambda < -1$ then

$$\|\vec{U}^{n+1}\|_\infty \leq \frac{1}{2}\big((-1-a\lambda)+(1-a\lambda)\big)\,\|\vec{U}^n\|_\infty = -a\lambda\|\vec{U}^n\|_\infty = \frac{-a}{h}k\,\|\vec{U}^n\|_\infty$$

again it can not be bounded by $(1+Kk)\|\vec{U}^n\|_\infty$ $\Rightarrow$ unstable

Conclusion: the socheme is stable only when $|a\lambda| \leq 1$.

#2 We first compute the following

① $\dfrac{u(t+k,x)-u(t,x)}{k} - u_t(t,x) = \dfrac{k u_t(t,x)+O(k^2)}{k} - u_t(t,x) = O(k)$

② $\dfrac{u(t,x+2h)-3u(t,x+h)+3u(t,x)-u(t,x-h)}{h^3} - u_{xxx}(t,x)$

$$= \frac{1}{h^3}\Big[\Big(u(t,x)+2h\,u_x(t,x)+\frac{(2h)^2}{2!}u_{xx}(t,x)+\frac{(2h)^3}{3!}u_{xxx}(t,x)+O(h^4)\Big)$$
$$-3\Big(u(t,x)+h\,u_x(t,x)+\frac{h^2}{2!}u_{xx}(t,x)+\frac{h^3}{3!}u_{xxx}(t,x)+O(h^4)\Big)$$
$$+3u(t,x)$$
$$-\Big(u(t,x)-h\,u_x(t,x)+\frac{h^2}{2!}u_{xx}(t,x)-\frac{h^3}{3!}u_{xxx}(t,x)+O(h^4)\Big)\Big]$$
$$-u_{xxx}(t,x)$$

$$= \frac{1}{h^3}\Big[\cancel{(1-3+3-1)}^{\,0}\,u(t,x)+\cancel{(2h-3h+h)}^{\,0}\,u_x(t,x)$$
$$+\cancel{\Big(\frac{4}{2}h^2-\frac{3}{2}h^2-\frac{1}{2}h^2\Big)}^{\,0}u_{xx}(t,x)$$
$$+\Big(\frac{8h^3}{6}-\frac{3h^3}{6}+\frac{h^3}{6}\Big)u_{xxx}(t,x)+O(h^4)\Big]$$
$$-u_{xxx}(t,x)$$

$$= \big[u_{xxx}(t,x)+O(h)\big]-u_{xxx}(t,x)$$
$$= O(h)$$

Then $\quad P_{k,h}u - R_{k,h}Pu$

$$= \Big[\frac{u(t+k,x)-u(t,x)}{k}+a\,\frac{u(t,x+2h)-3u(t,x+h)+3u(t,x)-u(t,x-h)}{h^3}\Big]$$
$$-\big[u_t(t,x)+a\,u_{xxx}(t,x)\big)$$

$$= O(k)+O(h)$$

#3 (Consistency) $P_{k,h}u - R_{k,h}Pu$

$$= \left[\frac{1}{2}\left(\frac{u(t+k,x)-u(t,x)}{k} + \frac{u(t+k,x+h)-u(t,x+h)}{k}\right)\right.$$
$$\left. + \frac{a}{2}\left(\frac{u(t+k,x+h)-u(t+k,x)}{h} + \frac{u(t,x+h)-u(t,x)}{h}\right)\right]$$
$$- [u_t + a u_x](t,x)$$

$$= \left[\frac{1}{2}\left(u_t(t,x) + u_t(t,x+h) + O(k)\right)\right.$$
$$\left. + \frac{a}{2}\left(u_x(t+k,x) + u_x(t,x) + O(h)\right)\right]$$
$$- [u_t(t,x) + a u_x(t,x)]$$

$$= \left[\frac{1}{2}\left(u_t(t,x) + (u_t(t,x) + O(h)) + O(k)\right)\right.$$
$$\left. + \frac{a}{2}\left((u_x(t,x) + O(k)) + u_x(t,x) + O(h)\right)\right]$$
$$- [u_t(t,x) + a u_x(t,x)]$$
$$= [u_t(t,x) + a u_x(t,x) + O(h) + O(k)]$$
$$- [u_t(t,x) + a u_x(t,x)]$$
$$= O(h) + O(k) \xrightarrow{h,k\to 0} 0$$

There for the scheme is consistent

(Stability) Use the von Neumann analysis, one has

$$\hat{V}^{n+1} = g(h,k,\theta)\hat{V}^n \quad \text{where} \quad g(h,k,\theta) = \frac{(1+a\lambda)+(1-a\lambda)e^{i\theta}}{(1-a\lambda)+(1+a\lambda)e^{i\theta}}$$

$$\Rightarrow g(h,k,\theta) = \frac{[\cdots]e^{-\frac{1}{2}i\theta}}{[\cdots]e^{-\frac{1}{2}i\theta}} = \frac{(1+a\lambda)e^{-\frac{1}{2}i\theta} + (1-a\lambda)e^{\frac{1}{2}i\theta}}{(1-a\lambda)e^{-\frac{1}{2}i\theta} + (1+a\lambda)e^{\frac{1}{2}i\theta}}$$

$$= \frac{(1+a\lambda)[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}] + (1-a\lambda)[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}]}{(1-a\lambda)[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}] + (1+a\lambda)[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}]} = \frac{2\cos\frac{\theta}{2} - 2ia\lambda\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2} + 2ia\lambda\sin\frac{\theta}{2}}$$

$$\Rightarrow |g(h,k,\theta)| = \frac{\sqrt{4\cos^2\frac{\theta}{2} + 4a^2\lambda^2\sin^2\frac{\theta}{2}}}{\sqrt{4\cos^2\frac{\theta}{2} + 4a^2\lambda^2\sin^2\frac{\theta}{2}}} = 1 \Rightarrow \text{unconditionally stable.}$$

#4 The scheme is

$$\frac{U_m^{n+1}-U_m^n}{k} + a\,\frac{s(U_{m+1}^{n+1}-U_{m-1}^{n+1})+(1-s)(U_{m+1}^{n}-U_{m-1}^{n})}{2h} = 0$$

$$\Rightarrow \quad U_m^{n+1} + s\,\frac{a\lambda}{2}\left(U_{m+1}^{n+1}-U_{m-1}^{n+1}\right) = U_m^n - (1-s)\frac{a\lambda}{2}\left(U_{m+1}^{n}-U_{m-1}^{n}\right)$$

$$\Rightarrow \quad \left[1+s\frac{a\lambda}{2}(e^{i\theta}-e^{-i\theta})\right]\hat{U}^{n+1} = \left[1-(1-s)\frac{a\lambda}{2}(e^{i\theta}-e^{-i\theta})\right]\hat{U}^n$$

$$\Rightarrow \quad \hat{U}^{n+1} = g(h,k,\theta)\,\hat{U}^n \quad \text{where}$$

$$g(h,k,\theta) = \frac{1+(1-s)\frac{a\lambda}{2}(e^{i\theta}-e^{-i\theta})}{1+s\frac{a\lambda}{2}(e^{i\theta}-e^{-i\theta})} = \frac{1-(1-s)\,i\,a\lambda\sin\theta}{1+s\,i\,a\lambda\sin\theta}$$

$$\Rightarrow \quad |g|^2 = \frac{1+(1-s)^2(a\lambda\sin\theta)^2}{1+s^2(a\lambda\sin\theta)^2}$$

Case I If $\frac{1}{2} \le s \le 1$, then we have $1-s \le s$ and consequently $|g|^2 \le 1$. The scheme is unconditionally stable

Case 2 If $0 \le s < \frac{1}{2}$, then

① Under the refinement path $\frac{k}{h}$ = constant, i.e. λ = constant,

$$|g|^2 = 1 + \frac{1-2s}{1+s^2(a\lambda\sin\theta)^2}(a\lambda\sin\theta)^2 > 1$$

and it can not be bounded by $(1+Kk)^2$

Hence the scheme is unstable

② If we use the refinement path $\frac{k}{h^2} = \mu$ = const. then

$$|g|^2 = 1 + \frac{1-2s}{1+s^2(a\lambda\sin\theta)^2}(a\lambda\sin\theta)^2 \le 1+(1-2s)a^2\lambda^2$$

$$= 1+(1-2s)a^2\mu k \le (1+Kk)^2 \Rightarrow \text{stable}$$