#3. Note that

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{[u + ku_t + O(k^2) - u](t_n, x_m)}{k} = u_t(t_n, x_m) + O(k)$$

$$\frac{u_{m+1}^{n+1} - u_m^{n+1}}{h} = \frac{u(t_{n+1}, x_{m+1}) - u(t_{n+1}, x_m)}{h}$$

$$= \frac{[u(t_{n+1}, x_m) + h\, u_x(t_{n+1}, x_m) + O(h^2)] - u(t_{n+1}, x_m)}{h}$$

$$= u_x(t_{n+1}, x_m) + O(h)$$

$$= [u_x(t_n, x_m) + k\, u_{xt}(t_n, x_m) + O(k^2)] + O(h)$$

$$\frac{u_m^n - u_{m-1}^n}{h} = \frac{[u - (u - hu_x + O(h^2))](t_n, x_m)}{h} = u_x(t_n, x_m) + O(h)$$

Therefore

$$[Lu - L_{h,k}u](t_n, x_m) = [u_t + au_x](t_n, x_m)$$
$$- \left[u_t + O(k) + \frac{a}{2}(u_x + ku_{xt} + O(k^2) + u_x + O(h))\right](t_n, x_m)$$

$$= \left[(\cancel{u_t + au_x}) - (\cancel{u_t + au_x} + \frac{a}{2}k u_{xt} + O(k) + O(h))\right](t_n, x_m)$$

$$= O(k) + O(h)$$

#4 Apply Fourier transform to

$$u_t + u_x + bu = 0$$

gives

$$\hat{u}_t + i\xi \hat{u} + b\hat{u} = 0$$

$$\Rightarrow \quad \hat{u}_t = -(i\xi + b)\hat{u}$$

Together with the initial condition $u(0,x) = u_0(x)$, one has

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)\, e^{-(i\xi+b)t}$$

Hence

$$\int_{-\infty}^{\infty} |u(t,x)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(t,\xi)|^2 d\xi$$

$$= \int_{-\infty}^{\infty} |\hat{u}_0(\xi)\, e^{-(i\xi + bt)}|^2 d\xi$$

$$= \int_{-\infty}^{\infty} |\hat{u}_0(\xi)|^2 \underbrace{|e^{-i\xi t}|^2}_{1} |e^{-bt}|^2 d\xi$$

$$= e^{-2bt} \int_{-\infty}^{\infty} |\hat{u}_0(\xi)|^2 d\xi$$

when $0 \le t \le T$ $\quad \le \underbrace{\max(e^0, e^{-2bT})}_{= C_T} \int_{-\infty}^{\infty} |u_0(x)|^2 dx$