Math 5543, Exam 2, Nov. 24, 2013

Please turn in your exam to the main office (MSCS 401) by 5pm in Nov. 24.

1. (10 points) A second-order accurate approximation to the Neumann boundary condition $\frac{\partial u}{\partial x}(t, x_0) = 0$ on the left boundary $x = x_0$ is given by

$$\frac{-3U_0^n + 4U_1^n - U_2^n}{2h} = 0.$$

   Similarly, design a second-order accurate approximation to $\frac{\partial u}{\partial x}(t, x_M) = 0$ on the right boundary $x = x_M$, and prove that the approximation error is of $O(h^2)$.

2. (10 points) Show that the forward-time upwind scheme for $u_t + au_x = bu_{xx}$, with $a > 0$, $b > 0$, is the same as the forward-time central-space scheme for $u_t + au_x = (b + \frac{ah}{2})u_{xx}$. (The extra diffusion $\frac{ah}{2}u_{xx}$ is often called "artificial diffusion". This indeed explains why upwinding scheme works well for convection dominant problems – because it adds extra diffusion (dissipation) to smooth out oscillations in the numerical solution.)

3. (15 points) Show that the phase error of the backward-time central-space scheme for $u_t + au_x = 0$ is

$$\frac{a(h\xi)^2}{6}(1 + 2a^2\lambda^2) + O(h\xi)^4$$

4. (15 points) Consider the semi-implicit, upwind scheme for $u_t + au_x = bu_{xx}$, with $a > 0$, $b > 0$,

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_m^n - U_{m-1}^n}{h} = b\frac{U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}}{h^2}$$

   Note this scheme is explicit for the convection part u_x and implicit for the diffusion part u_{xx}. Derive a sufficient and necessary condition for this scheme to be stable.