#1. Define an approximation

$$\frac{3U_m^n - 4U_{m-1}^n + U_{m-2}^n}{2h} = 0$$

to $\frac{\partial u}{\partial x}(t_n, x_m) = 0$.

We use Taylor expansion to compute the order of accuracy. For simplicty, all $u, u_x, u_{xx}, \cdots$, are ~~evaluated~~ centered at (t_n, x_m). Then

$$\frac{\partial u}{\partial x}(t_n, x_m) - \frac{3u(t_n, x_m) - 4u(t_n, x_{m-1}) + u(t_n, x_{m-2})}{2h}$$

$$= u_x - \frac{3u - 4\left[u - hu_x + \frac{h^2}{2!}u_{xx} + O(h^3)\right] + \left[u - 2hu_x + \frac{(2h)^2}{2!}u_{xx} + O(h^3)\right]}{2h}$$

(all functions evaluated at (t_n, x_m))

$$= u_x - \frac{2hu_x + O(h^3)}{2h}$$

$$= O(h^2)$$

2 The FT-upwinding scheme is

$$\frac{U_m^{n+1}-U_m^n}{k} + a\frac{U_m^n - U_{m-1}^n}{h} = b\,\delta^2 U_m^n$$

$$\Rightarrow \quad \frac{U_m^{n+1}-U_m^n}{k} + a\left[\frac{U_{m+1}^n - U_{m-1}^n}{2h} - \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{2h}\right] = b\,\delta^2 U_m^n$$

$$\Rightarrow \quad \frac{U_m^{n+1}-U_m^n}{k} + a\,\delta^0 U_m^n - \frac{ah}{2}\,\delta^2 U_m^n = b\,\delta^2 U_m^n$$

$$\Rightarrow \quad \frac{U_m^{n+1}-U_m^n}{k} + a\,\delta^0 U_m^n = \left(b + \frac{ah}{2}\right)\delta^2 U_m^n$$

Which is exactly the FT-CS scheme for

$$u_t + a u_x = \left(b + \frac{ah}{2}\right) u_{xx}$$

#3. The scheme is

$$\frac{U_m^{n+1} - U_m^n}{k} + a\,\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2h} = 0$$

$$\Rightarrow \quad g - 1 + \frac{a\lambda}{2} g \left(e^{i\theta} - e^{-i\theta}\right) = 0$$

$$\Rightarrow \quad g = \frac{1}{1 + a\lambda i \sin\theta} = \frac{1 - a\lambda i \sin\theta}{1 + (a\lambda \sin\theta)^2}$$

$$= \frac{1}{1 + (a\lambda\sin\theta)^2} - i\,\frac{a\lambda\sin\theta}{1 + (a\lambda\sin\theta)^2}$$

The phase speed is

$$\alpha(h\xi) = \frac{\arctan\left(-\frac{\operatorname{Im} g}{\operatorname{Re} g}\right)}{k\xi} = \frac{\arctan(a\lambda\sin\theta)}{\lambda h\xi}$$

$$= \frac{1}{\lambda\theta}\left[a\lambda\sin\theta - \frac{1}{3}(a\lambda\sin\theta)^3 + O(\sin\theta)^5\right]$$

$$= \frac{1}{\lambda\theta}\left[a\lambda\left(\theta - \frac{1}{3!}\theta^3 + O(\theta)^5\right) - \frac{(a\lambda)^3}{3}\left(\theta - \frac{1}{3!}\theta^3 + O(\theta)^5\right)^3 + O(\theta)^5\right]$$

$$= \frac{1}{\lambda\theta}\left[a\lambda\theta - \frac{1}{6}a\lambda\theta^3 - \frac{(a\lambda)^3}{3}\theta^3 + O(\theta)^5\right]$$

$$= a - \frac{a\theta^2}{6}\left(1 + 2a^2\lambda^2\right) + O(\theta)^4$$

$\Rightarrow$ The phase error is

$$a - \alpha(h\xi) = \frac{a\theta^2}{6}\left(1 + a^2\lambda^2\right) + O(\theta)^4$$

$$= \frac{a(h\xi)^2}{6}\left(1 + 2a^2\lambda^2\right) + O(h\xi)^4$$

4 We use the von Neumann analysis

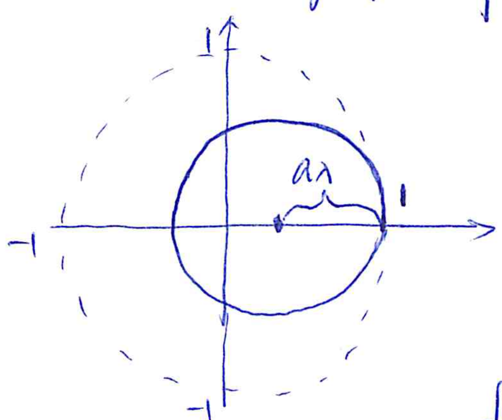
$$g - 1 + a\lambda\left(1 - e^{-i\theta}\right) = b\mu g\left(e^{i\theta} - 2 + e^{-i\theta}\right)$$

$$\Rightarrow \quad g = \frac{1 - a\lambda + a\lambda e^{-i\theta}}{1 + 4b\mu \sin^2\frac{\theta}{2}}$$

Since $b>0$, $\mu>0$, one clearly has

$$|g| = \frac{|1 - a\lambda + a\lambda e^{-i\theta}|}{|1 + 4b\mu\sin^2\frac{\theta}{2}|} \le |1 - a\lambda + a\lambda e^{-i\theta}|$$

Note the graph of $1 - a\lambda + a\lambda e^{-i\theta}$ looks like



There fore (Since $a>0$)

$|g| \le 1$ if $a\lambda \le 1$

So ~~the~~ a sufficient condition for the scheme to be stable is

$$a\lambda \le 1$$

Next, we show this condition is also necessary. If $a\lambda > 1$, then at $\theta = \pi$ one has

$$|g(0)| = \left|\frac{1 - a\lambda + a\lambda e^{-i\pi}}{1 + 4b\mu\sin^2\pi}\right| = \left|\frac{1-2a\lambda}{1}\right| = 2a\lambda - 1 > 1$$

$\Rightarrow$ the scheme is not stable if $a\lambda > 1$