Math 5543, Exam 1, Oct. 6, 2013

1. (10 points) Show that the initial value problem for the equation

$$u_t + au_x - b^2 u_{xx} + cu_{xxx} = 0,$$

where a, b, c are real numbers, is well posed.

2. (10 points) Discuss the stability of the backward-time central-space scheme for $u_t + au_x = f$ using the von Neumann analysis.

3. (15 points) Determine the order of accuracy for the following scheme

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{-3U_m^n + 4U_{m+1}^n - U_{m+2}^n}{2h} = f_m^n$$

for solving $u_t + au_x = f$. (This scheme is centered at $(t_n,\, x_m)$.)

4. (15 points) Consider the initial boundary value problem

$$\begin{cases} u_t + 2u_x = 1 & \text{for } 0 \le x \le 1 \text{ and } t \ge 0 \\ u(0,x) = \sin(2\pi x) & \text{for } 0 \le x \le 1 \\ u(t,0) = 0 & \text{for } t \ge 0 \end{cases}$$

Apply the backward-time backward-space scheme to this problem with $h = k = 0.25$. Write a linear system for computing the values of the grid function at $t_1 = 0.25$, i.e., find matrix B and right-hand side vector $\mathbf{r}$ such that

$$B \begin{bmatrix} U_0^1 \\ U_1^1 \\ U_2^1 \\ U_3^1 \\ U_4^1 \end{bmatrix} = \mathbf{r}.$$