#1 Apply the (continuous) Fourier transform, we have

$$\hat{u}_t + \left[a i\xi - b^2 (i\xi)^2 + c(i\xi)^3\right]\hat{u} = 0$$

$$\Rightarrow \hat{u}_t = -\left[a i\xi + b^2\xi^2 - c i\xi^3\right]\hat{u}$$

$$= \left[-b^2\xi^2 - i(a\xi - c\xi^3)\right]\hat{u}$$

Solve the ODE, one gets

$$\hat{u} = \hat{u}_0 \, e^{[-b^2\xi^2 - i(a\xi - c\xi^3)]t}$$

$$\Rightarrow \|u\|_{L^2}^2 = \|\hat{u}\|_{L^2}^2 = \int_{-\infty}^{\infty} \left|\hat{u}_0 \, e^{[-b^2\xi^2 - i(a\xi - c\xi^3)]t}\right|^2 d\xi$$

$$= \int_{-\infty}^{\infty} |\hat{u}_0|^2 \left|e^{-2b^2\xi^2 t}\right| \cdot \cancelto{1}{\left|e^{-2i(a\xi - c\xi^3)t}\right|} \, d\xi$$

(Note that $\left|e^{-2b^2\xi^2 t}\right| \leq 1$ for all $t \geq 0$)

$$\leq \int_{-\infty}^{\infty} |\hat{u}_0|^2 \, d\xi$$

$$= \|\hat{u}_0\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

Hence the scheme is well-posed.

#2 The scheme is

$$\frac{U_m^{n+1} - U_m^n}{k} + a\,\frac{U_{m+1}^{n+1} - U_{m-1}^{n+1}}{2h} = f_m^{n+1}.$$

By Von Neumann analysis. we have (set $f \equiv 0$)

$$U_m^{n+1} + \frac{a\lambda}{2} U_{m+1}^{n+1} - \frac{a\lambda}{2} U_{m-1}^{n+1} = U_m^n$$

$$\Rightarrow \quad \left(1 + \frac{a\lambda}{2} e^{i\theta} - \frac{a\lambda}{2} e^{-i\theta}\right) \hat{U}^{n+1} = \hat{U}^n$$

$$\Rightarrow \quad g(h,k,\theta) = \frac{1}{1 + \frac{a\lambda}{2}(e^{i\theta} - e^{-i\theta})}$$

$$= \frac{1}{1 + a\lambda i \sin\theta}$$

$$\Rightarrow \quad |g|^2 = \frac{1}{1 + (a\lambda \sin\theta)^2} \leq 1$$

$\Rightarrow$ the scheme is unconditionally stable.

#3 $P_{k,h}u - R_{k,h}Pu$

$$= \left[\frac{u(t+k,x) - u(t,x)}{k} + a\,\frac{-3u(t,x) + 4u(t,x+h) - u(t,x+2h)}{2h} \right] - \left[u_t(t,x) + a u_x(t,x) \right]$$

$$= \left[\frac{k u_t(t,x) + \frac{k^2}{2!} u_{tt}(t,x) + O(k^3)}{k} + a \cdot \frac{1}{2h} \left(-3u(t,x) + 4\left(u(t,x) + h u_x(t,x) + \frac{h^2}{2!} u_{xx}(t,x) + \frac{h^3}{3!} u_{xxx}(t,x) + O(h^4)\right) - \left(u(t,x) + (2h) u_x(t,x) + \frac{(2h)^2}{2!} u_{xx}(t,x) + \frac{(2h)^3}{3!} u_{xxx}(t,x) + O(h^4) \right) \right) \right] - \left[u_t(t,x) + a u_x(t,x) \right]$$

$$= \left[u_t(t,x) + O(k) + \frac{a}{2h}\left(2h u_x(t,x) + O(h^3)\right) \right] - \left[u_t(t,x) + a u_x(t,x) \right]$$

$$= O(k) + O(h^2)$$

#4 The scheme is $\dfrac{U_m^{n+1}-U_m^n}{k} + a\,\dfrac{U_m^{n+1}-U_{m-1}^{n+1}}{h} = f_m^{n+1}$

which implies, for this problem

$$\frac{U_m^1 - U_m^0}{0.25} + 2\,\frac{U_m^1 - U_{m-1}^1}{0.25} = 1 \qquad \text{for } m = 1, 2, \cdots 4$$

and for $m = 0$, one has by the boundary condition

$$U_0^1 = 0$$

Combine the above 5 equations gives

$$\begin{cases} U_0^1 = 0 \\ 3U_1^1 - 2U_0^1 = 0.25 + U_1^0 = 0.25 + \sin\frac{\pi}{2} = 1.25 \\ 3U_2^1 - 2U_1^1 = 0.25 + U_2^0 = 0.25 + \sin\pi = 0.25 \\ 3U_3^1 - 2U_2^1 = 0.25 + U_3^0 = 0.25 + \sin\frac{3\pi}{2} = -0.75 \\ 3U_4^1 - 2U_3^1 = 0.25 + U_4^0 = 0.25 + \sin 2\pi = 0.25 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} U_0^1 \\ U_1^1 \\ U_2^1 \\ U_3^1 \\ U_4^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.25 \\ 0.25 \\ -0.75 \\ 0.25 \end{bmatrix}$$