Homework 4 Due 12/5/2013

1. (6 points) Examine the stability of the Douglass-Rachford method (7.3.10).

2. (6 points) Examine the stability of the central-time central-space scheme for $u_{tt} - a^2 u_{xx} = 0$.

3. (8 points) (Problem 3, Numerical Analysis Comprehensive Exam, August, 2007) Let function $u(x, y)$ be the solution to the following Dirichlet problem (which means Poisson equation with Dirichlet boundary conditions):

$$\begin{cases} \nabla^2 u \triangleq u_{xx} + u_{yy} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \{(x, y) \mid |x| + |y| < 1\}$. Let the grid function v be defined on the uniform grid over $\bar{\Omega} \triangleq \{(x, y) \mid |x| + |y| \leq 1\}$ (the closure of Ω). In other words, v_{lm} denotes the value of v at a point $(x_l, y_m) = (hl, hm)$ in $\bar{\Omega}$, where h is the step size and $1/h$ is an integer. Denote ∇_h^2 to be the standard five-point Laplacian. Suppose that the grid function v is the solution to $\nabla_h^2 v = f$ over the interior of Ω, and $v_{lm} = 0$ on the boundary of Ω. Prove that there exists a constant $C_0 > 0$ independent of h and u such that

$$\|u - v\|_\infty \leq C_0 h^2 \|\partial^4 u\|_\infty.$$

Here the maximum norm $\|\cdot\|_\infty$ is taken on all grid points in Ω.