Homework 2 Due 10/1/2013

1. (4 points) Analyze the order of accuracy and stability of the Crank-Nicolson scheme.

2. (5 points) (NA comprehensive exam, Summer 2012) Consider the scheme

$$\frac{1}{2k}\left[(v_m^{n+1}+v_{m+1}^{n+1})-(v_m^n+v_{m+1}^n)\right]+\frac{a}{2h}\left[(v_{m+1}^{n+1}-v_m^{n+1})+(v_{m+1}^n-v_m^n)\right]=0$$

for the one-way wave equation $u_t + au_x = 0$. Here a is a real constant, v_m^n is the value of the grid function defined on $(x_m, t_n) = (mh, nk)$, for $m \in \mathbb{Z}$, $n \in \{0\} \cup \mathbb{Z}^+$. Prove the scheme is consistent and is stable in 2-norm for all values of $\lambda = k/h$.

3. (5 points) Consider the following fourth-order leap-frog scheme for $u_t + au_x = 0$:

$$\frac{u_m^{n+1}-u_m^{n-1}}{2k}+a\left(\frac{4}{3}\frac{u_{m+1}^n-u_{m-1}^n}{2h}-\frac{1}{3}\frac{u_{m+2}^n-u_{m-2}^n}{4h}\right)=0.$$

Analyze its order of accuracy and stability.

4. (6 points) The parasitic mode of the leap-frog scheme, if we are not careful, can cause troubles. An interesting example was given in "Group velocity in finite difference schemes" by L.N. Trefethen, SIAM Review Vol. 24, No. 2, 1982, pp 113-136. Below is (a slightly modified) version of this example.

Consider the one way wave equation defined on $-1 \le x \le 1$ and $0 \le t < \infty$, with inhomogeneous wave speed

$$u_t + au_x = 0, \qquad a = \begin{cases} 1 & \text{for } x \le 0 \\ 0.7 & \text{for } x > 0 \end{cases},$$

with initial and boundary conditions

$$u(0,x) = 0 \qquad \text{and} \qquad u(t,-1) = \sin 20t.$$

Physically, this can be viewed as a wave on a string driven by a periodic movement on the left end of the string. The wave moves to the right and there is an "interface" at $x = 0$ where the wave speed drops. Try to understand how the wave propagates using characteristic lines.

Simulate this problem using the leap-frog scheme, with the following specifications:

- Set $h = 0.01$ and $k = 0.005$;
- Use forward-time backward-space scheme at t_1;
- Use leap-frog scheme at t_n for $n > 1$. For the right boundary, use (3.4.1d). Read page 86 of the textbook, you can see that (3.4.1d) together with leap-frog scheme (with $|a\lambda| < 1$) is stable.
- When applying the schemes, use $a = 1$ for $x_m \le 0$ and $a = 0.7$ for $x_m > 0$.

Plot the solution at $t = 0.5$, 1, 1.5, 2. What do you observe?

Think about the following questions but they will not be graded. From the graph you should see at $t = 1$, the wave front reaches the interface $x = 0$. At this point, the finite difference scheme introduces a reflected parasite (the small oscillations) which travels to the left. You can try to make a movie in Matlab, which will better illustrate the wave propagation as well as the artificial oscillation introduced by the scheme. Note the true solution should not contain any oscillation. Is it possible to get ride of the numerical oscillations?