Project: Eigenvalues/Eigenfunctions of the Laplacian

Consider the rectangular domain $\Omega = (0, \pi) \times (0, \pi)$, and the eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It has infinitely many eigenvalues/eigenfunctions, which are given by

$$\lambda^{(k,l)} = k^2 + l^2, \qquad u^{(k,l)} = \sin kx \sin ly$$

for all positive intergers k and l.

Use a five-point scheme to discretize this problem on an $M \times N$ grid where $M = N$. The step size is $h = \frac{\pi}{M} = \frac{\pi}{N}$. This gives a discrete eigenvalue problem:

$$A\bar{u} = \bar{\lambda}\bar{u}$$

where $\bar{u}$ is a grid function taking values on all grid points (x_m, y_n). The discrete eigenvalues/eigenfunctions have the form

$$\bar{\lambda}^{(k,l)} = \frac{2}{h^2}(2 - \cos kh - \cos lh), \qquad \bar{u}^{(k,l)}_{m,n} = \sin kx_m \sin ly_n$$

for $k, l = 1, 2, \cdots, M$. This can be proved by direct substitution.

Consider the following questions. You team is expected to submit a project report containing both theoretical survey and numerical results, and to make a classroom presentation.

1. Does the discrete eigenvalue problem provide a good approximation to the continuous eigenvalue problem?

2. Write a Matlab program to compute the discrete eigenvalues/eigenfunctions. Check whether your solution satisfy the Courant Nodal Domain Theorem.

   Read the note by Naoki Saito about the Courant Nodal Domain Theorem at https://www.math.ucdavis.edu/ saito/courses/LapEig/lecpdf/lecture7.pdf

3. To make the numerical results more interesting, consider the problem

   $$-\Delta u + f(x, y)u = \lambda u,$$

   where $f(x, y) \geq 0$ for all $(x, y) \in \Omega$. You can test with different function $f(x, y)$.

   Read Spectral Method in Matlab by L.N. Trefethen, Chapter 9 for some numerical examples.