Homework assignment 4

1. (Numerical Analysis Comprehensive Exam, August 2006, #2) Consider the following finite difference scheme:

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+\frac{U_j^{n+1}-U_{j-1}^{n+1}}{\Delta x}=0, \quad n=0,\dots,N-1, \quad j=1,\dots,J,$$
$$U_0^n=0, \quad n=0,\dots,N,$$
$$U_j^0=u_0(j\Delta x), \quad j=0,\dots,J,$$

   where $\Delta t = 1/N$ and $\Delta x = 1/J$. Derive the exact condition for the scheme to be stable in the maximum norm.

2. Stability of the leapfrog scheme can also be examined by viewing it as a one-step scheme. The leapfrog scheme for $u_t + au_x = 0$ is

$$u_m^{n+1}=u_m^{n-1}-a\lambda(u_{m+1}^n-u_{m-1}^n),$$

   where $\lambda = k/h$. It can be written in the form:

$$\vec{u}^{n+1}=\vec{u}^{n-1}+A\vec{u}^n,$$

   where

$$A=\begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ & a\lambda & 0 & -a\lambda & & & \\ & & a\lambda & 0 & -a\lambda & & \\ & & & a\lambda & 0 & -a\lambda & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

   Hence we have

$$\begin{bmatrix}\vec{u}^{n+1}\\ \vec{u}^n\end{bmatrix}=\begin{bmatrix}A & I\\ I & 0\end{bmatrix}\begin{bmatrix}\vec{u}^{n}\\ \vec{u}^{n-1}\end{bmatrix}, \qquad \text{where } I \text{ is the identity matrix} \tag{1}$$

   The above scheme (1) can be viewed as a one-step scheme $\vec{w}^{n+1} = B\vec{w}^n$.

   (a) The Fourier transform of scheme (1) can be written as

$$\begin{bmatrix}\hat{u}^{n+1}\\ \hat{u}^n\end{bmatrix}=G(h\xi)\begin{bmatrix}\hat{u}^{n}\\ \hat{u}^{n-1}\end{bmatrix},$$

   where $G(h\xi)$ is a 2×2 matrix. Compute this matrix.

   (b) Compute the Jordan canonical form of $G(h\xi) = T\Lambda T^{-1}$. Here Λ is a 2×2 matrix whose diagonal components $\lambda_{1,2}$ are eigenvalues of $G(h\xi)$, and T is a 2×2 matrix whose columns are eigenvectors corresponding to $\lambda_{1,2}$. You need to discuss for different situations. Be careful when $G(h,\xi)$ has a repeated eigenvalue.

   (c) Use the Jordan form to find the exact condition for the leapfrog scheme to be stable.

3. Textbook 5.1.7, 5.2.2, 6.1.4.