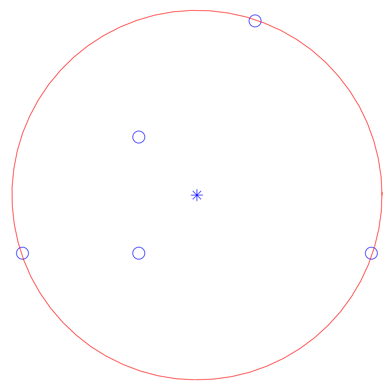
Math 4553, Homework 5, Due on 4/25/2014

1. (10 points) Consider the smallest circle problem: given n points $p_i\,(x_i,\, y_i)$ on a 2D plane, find the smallest circle that contains all these points. For example, in the following graph, five points (blue circles) are given. The red circle, which is centered at the blue star, is the smallest circle that encloses all five points.



Denote the $2 \times n$ matrix P by

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

This smallest circle problem can be formulated as a quadratic programming problem (see *The Minimum Covering Sphere Problem* by D.J. Elzinga and D.W. Hearn, Management Science, Vol. 19, page 96-104, 9172)

$$\begin{aligned} \text{optimization variables} \quad & \mathbf{c} = (c_1,\, c_2, \ldots, c_n) \\ \min \quad & f(\mathbf{c}) = \mathbf{c}^T P^T P \mathbf{c} - \sum_{i=1}^{n} c_i (x_i^2 + y_i^2) \\ \text{subject to} \quad & \sum_{i=1}^{n} c_i = 1 \\ & c_i \geq 0, \text{ for } i = 1, \ldots, n \end{aligned}$$

If $\bar{\mathbf{c}}$ is the solution of this quadratic programming problem, then the center of the circle is located at $p\,(x,\, y)$ where

$$\begin{cases} x = \bar{c}_1 x_1 + \bar{c}_2 x_2 + \cdots \bar{c}_n x_n \\ y = \bar{c}_1 y_1 + \bar{c}_2 y_2 + \cdots \bar{c}_n y_n \end{cases}$$

and the radius of the circle is given by $r = \sqrt{-f(\bar{\mathbf{c}})}$.

Given five points:

$$p_1\,(3,\, -3), \quad p_2\,(8,\, 2), \quad p_3\,(0,\, 6), \quad p_4\,(6,\, 5), \quad p_5\,(-1,\, 0).$$

Form the quadratic programming problem and use the Lemke's method to compute the center and radius of the smallest circle that contains all five points.

2. (10 points) A company wants to build a rectangular water tank without the top lid, which holds exactly 400 m^3 of water. The tank will be put inside a building with a height clearance of 8 meters. Suppose the bottom and sides of the water tank are made from the same material. Formulate an optimization problem to help the company to minimize the material costs. Then, write the KKT conditions for the optimization problem (you do not need to solve it).