Math 4553, Exam II, Apr. 9, 2012, Solution

1. First, rewrite the "$\leq$" type constraints of the primal problem into the "$\geq$" type constraints. We get

$$\begin{aligned}
\text{min} \quad & f = 3x_1 + 4x_2 + 5x_3 + 6x_4 \\
\text{subject to} \quad & -x_1 + x_2 - 4x_3 - 6x_4 \geq 0 \\
& x_1 + 2x_2 + 2x_3 + x_4 = 9 \\
& 5x_1 - 6x_2 - 7x_3 + 8x_4 \geq 3 \\
& x_2 \geq 0,\ x_3 \geq 0 \\
& x_1,\ x_4 \text{ are free variables}
\end{aligned}$$

Clearly

$$A = \begin{pmatrix} -1 & 1 & -4 & -6 \\ 1 & 2 & 2 & 1 \\ 5 & -6 & -7 & 8 \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 9 \\ 3 \end{pmatrix}$$

Therefore, the dual problem is

$$\begin{aligned}
\text{max} \quad & g = 9u_2 + 3u_3 \\
\text{subject to} \quad & -u_1 + u_2 + 5u_3 = 3 \\
& u_1 + 2u_2 - 6u_3 \leq 4 \\
& -4u_1 + 2u_2 - 7u_3 \leq 5 \\
& -6u_1 + u_2 + 8u_3 = 6 \\
& u_1 \geq 0,\ u_3 \geq 0 \\
& u_2 \text{ is free}
\end{aligned}$$

2. (a) The KKT conditions are

$$\begin{aligned}
\mathbf{0} \leq \mathbf{x} \perp Q\mathbf{x} - A^t\mathbf{u} + \mathbf{p} \geq \mathbf{0} \\
\mathbf{0} \leq \mathbf{u} \perp A\mathbf{x} - \mathbf{b} \geq \mathbf{0}
\end{aligned}$$

(b) By the KKT condition, we have

$$\begin{aligned}
& x_1(x_1 - x_2 - u_1 + 4) = 0 \\
& x_2(-x_1 + x_2 - u_1 - 1) = 0 \\
& u_1(x_1 + x_2 - 2) = 0
\end{aligned}$$

Plug in $x_2 = 2$ and $u_1 = 1$ into the second equation, we have

$$2(-x_1 + 2 - 1 - 1) = 0$$

which implies $x_1 = 0$. The minimum value of f is

$$f = \frac{1}{2}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$$

3. Note the objective function can be written into

$$f = \frac{1}{2}\mathbf{x}^t \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 & -1 \end{bmatrix} \mathbf{x}$$

Therefore, matrix Q is

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Now let us compute the eigenvalues of Q. Clearly

$$\det(\lambda I - Q) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

The eigenvalues are 1 and 3, both are positive. Hence the quadratic programming problem is convex.

To use the Lemke's method, we first notice that

$$M = \begin{bmatrix} Q & -A^t \\ A & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} \mathbf{p} \\ -\mathbf{b} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ -1 \end{bmatrix}$$

Therefore, the initial tableau is

```
>> T = lemketbl([2,1,-1,-2;1,2,-1,0;1,1,0,0;2,0,0,0], [-1,-1,-2,-1]);
                 z1          z2          z3          z4          1
         -----------------------------------------------------------
 w1 = |      2.0000      1.0000     -1.0000     -2.0000     -1.0000
 w2 = |      1.0000      2.0000     -1.0000      0.0000     -1.0000
 w3 = |      1.0000      1.0000      0.0000      0.0000     -2.0000
 w4 = |      2.0000      0.0000      0.0000      0.0000     -1.0000
```

Now, add the z_0 column:

```
>> T = addcol(T,[1,1,1,1]','z0',5);
                 z1          z2          z3          z4          z0          1
         -----------------------------------------------------------------------
 w1 = |      2.0000      1.0000     -1.0000     -2.0000      1.0000     -1.0000
 w2 = |      1.0000      2.0000     -1.0000      0.0000      1.0000     -1.0000
 w3 = |      1.0000      1.0000      0.0000      0.0000      1.0000     -2.0000
 w4 = |      2.0000      0.0000      0.0000      0.0000      1.0000     -1.0000
```

Then,, pivot z_0 with the row corresponding to the most negative entry in the last column, that is, row 3

```
>> T = ljx(T,3,5);
                 z1          z2          z3          z4          w3          1
         -----------------------------------------------------------------------
 w1 = |      1.0000      0.0000     -1.0000     -2.0000      1.0000      1.0000
 w2 = |      0.0000      1.0000     -1.0000      0.0000      1.0000      1.0000
 z0 = |     -1.0000     -1.0000     -0.0000     -0.0000      1.0000      2.0000
 w4 = |      1.0000     -1.0000      0.0000      0.0000      1.0000      1.0000
```

The tableau is now almost-complementary. This completes Phase I of the Lemke's method. Next, we start Phase II. Pivot z_3 with w_1

```
>> T = ljx(T,1,3);
              z1          z2          w1          z4          w3           1
       -----------------------------------------------------------------
 z3 = |     1.0000      0.0000     -1.0000     -2.0000      1.0000      1.0000
 w2 = |    -1.0000      1.0000      1.0000      2.0000      0.0000      0.0000
 z0 = |    -1.0000     -1.0000      0.0000     -0.0000      1.0000      2.0000
 w4 = |     1.0000     -1.0000     -0.0000      0.0000      1.0000      1.0000
```

Pivot z_1 with w_2

```
>> T = ljx(T,2,1);
              w2          z2          w1          z4          w3           1
       -----------------------------------------------------------------
 z3 = |    -1.0000      1.0000      0.0000      0.0000      1.0000      1.0000
 z1 = |    -1.0000      1.0000      1.0000      2.0000      0.0000      0.0000
 z0 = |     1.0000     -2.0000     -1.0000     -2.0000      1.0000      2.0000
 w4 = |    -1.0000      0.0000      1.0000      2.0000      1.0000      1.0000
```

Pivot z_2 with z_0

```
>> T = ljx(T,3,2);
              w2          z0          w1          z4          w3           1
       -----------------------------------------------------------------
 z3 = |    -0.5000     -0.5000     -0.5000     -1.0000      1.5000      2.0000
 z1 = |    -0.5000     -0.5000      0.5000      1.0000      0.5000      1.0000
 z2 = |     0.5000     -0.5000     -0.5000     -1.0000      0.5000      1.0000
 w4 = |    -1.0000     -0.0000      1.0000      2.0000      1.0000      1.0000
```

Now the tableau is complementary and we have

$$\begin{array}{ccccc} z_0 = 0, & z_1 = 1, & z_2 = 1, & z_3 = 2, & z_4 = 0 \\ w_0 = 0, & w_1 = 0, & w_2 = 0, & w_3 = 0, & w_4 = 1 \end{array}$$

therefore, $x_1 = z_1 = 1$ and $x_2 = z_2 = 1$, and the minimum value of f is 1.