Solution to Exam I

1. (a) Let x_i, $i = 1, \ldots, 5$, be the ammounts of investments on different types of stocks, then the objective function is

$$\max \qquad f = 0.12x_1 + 0.09x_2 + 0.05x_3 + 0.08x_4 + 0.04x_5$$

and the constraints are

$x_1 + x_2 + x_3 + x_4 + x_5 = 1,000,000$ (total ammount)

$x_2 \leq 25,000$ (precious metal)

$x_1 + x_3 \geq 30,000$ (medicine plus computer harware)

$\dfrac{3.2x_1 + 1.8x_2 + 1.6x_3 + 2.1x_4 + 1.4x_5}{1,000,000} \leq 2.0$ (risk factor)

$x_i \geq 0,\ i = 1, \ldots, 5$

Another way to solve the problem is to set x_i to be the percentage of investments. Then the objective function is

$$\max \qquad f = 1,000,000 \times (0.12x_1 + 0.09x_2 + 0.05x_3 + 0.08x_4 + 0.04x_5)$$

and the constraints are

$x_1 + x_2 + x_3 + x_4 + x_5 = 1$ (total percentage = 100%)

$1,000,000x_2 \leq 25,000$ (precious metal)

$1,000,000(x_1 + x_3) \geq 30,000$ (medicine plus computer harware)

$3.2x_1 + 1.8x_2 + 1.6x_3 + 2.1x_4 + 1.4x_5 \leq 2.0$ (risk factor)

$x_i \geq 0,\ i = 1, \ldots, 5$

(b) In part (b) and (c), we will only show the solution when x_i are set to be the amounts. The solution for x_i to be the percentage of investments is similar.

$$\begin{array}{rl} \min & f = -0.12x_1 - 0.09x_2 - 0.05x_3 - 0.08x_4 - 0.04x_5 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 + x_5 \geq 1,000,000 \\ & -x_1 - x_2 - x_3 - x_4 - x_5 \geq -1,000,000 \\ & -x_2 \geq -25,000 \\ & x_1 + x_3 \geq 30,000 \\ & -3.2x_1 - 1.8x_2 - 1.6x_3 - 2.1x_4 - 1.4x_5 \geq -2,000,000 \\ & x_i \geq 0,\ i = 1, \ldots, 5 \end{array}$$

(c) The dual problem is

$$\begin{array}{rl} \max & g = 1,000,000u_1 - 25,000u_2 + 30,000u_3 - 2,000,000u_4 \\ \text{subject to} & u_1 + u_3 - 3.2u_4 \leq -0.12 \\ & u_1 - u_2 - 1.8u_4 \leq -0.09 \\ & u_1 + u_3 - 1.6u_4 \leq -0.05 \\ & u_1 - 2.1u_4 \leq -0.08 \\ & u_1 - 1.4u_4 \leq -0.04 \\ & u_2,\ u_3,\ u_4,\ \geq 0,\ u_1 \text{ free} \end{array}$$

2. Since the problem is given in non-standard form, we will apply scheme II first. The initial tableau is

```
             x1          x2          x3          1
       ---------------------------------------------
y1 = |     2.0000     -1.0000      1.0000     -5.0000
y2 = |     1.0000      1.0000      0.0000    -10.0000
       ---------------------------------------------
f  = |     2.0000     -1.0000      2.0000      0.0000
```

Exchange y_1 and x_3, we have

```
             x1          x2          y1          1
       ---------------------------------------------
x3 = |    -2.0000      1.0000      1.0000      5.0000
y2 = |     1.0000      1.0000      0.0000    -10.0000
       ---------------------------------------------
f  = |    -2.0000      1.0000      2.0000     10.0000
```

Now $y_1 = 0$, we can delete the y_1-column. Since x_3 is a free variable, we will move it to the bottom. The new tableau is

```
             x1          x2          1
       ---------------------------------
y2 = |     1.0000      1.0000    -10.0000
       ---------------------------------
f  = |    -2.0000      1.0000     10.0000
x3 = |    -2.0000      1.0000      5.0000
```

This tableau is not feasible. Next, we need to apply phase I to get a starting basic feasible solution. Add artificial variable x_0 and objective function f_0,

```
             x1          x2          x0          1
       ---------------------------------------------
y2 = |     1.0000      1.0000      1.0000    -10.0000
       ---------------------------------------------
f  = |    -2.0000      1.0000      0.0000     10.0000
x3 = |    -2.0000      1.0000      0.0000      5.0000
f0 = |     0.0000      0.0000      1.0000      0.0000
```

Apply a special pivoting which exchanges y_2 with x_0,

```
             x1          x2          y2          1
       ---------------------------------------------
x0 = |    -1.0000     -1.0000      1.0000     10.0000
```

```
        ----------------------------------------
f   = |    -2.0000     1.0000     0.0000    10.0000
x3  = |    -2.0000     1.0000     0.0000     5.0000
f0  = |    -1.0000    -1.0000     1.0000    10.0000
```

Now try to minimize f_0. Exchange x_0 with x_1,

```
             x0         x2         y2          1
        ----------------------------------------
x1  = |    -1.0000    -1.0000     1.0000    10.0000
        ----------------------------------------
f   = |     2.0000     3.0000    -2.0000   -10.0000
x3  = |     2.0000     3.0000    -2.0000   -15.0000
f0  = |     1.0000     0.0000     0.0000     0.0000
```

The minimum of f_0 is reached and $f_0 = 0$. This completes phase I. Delete x_0-column and f_0-row,

```
             x2         y2          1
        ------------------------------
x1  = |    -1.0000     1.0000    10.0000
        ------------------------------
f   = |     3.0000    -2.0000   -10.0000
x3  = |     3.0000    -2.0000   -15.0000
```

The tableau is now feasible. We can start phase II.

Notice immediately that the tableau is unbounded, since no pivot row can be chosen. Indeed, if we let $y_2 = \lambda \geq 0$, then

$$x_1 = \lambda + 10 \geq 0, \qquad x_2 = 0, \qquad x_3 = -2\lambda - 15 \text{ (free variable)}$$

and

$$f = -2\lambda - 10.$$

As λ goes to $+\infty$, the value of f goes to $-\infty$.