Math 4553, Solution to Homework 6

1. Clearly

$$P = \begin{bmatrix} 1 & 2 & 3 & 4 & 2 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix}$$

and the optimization problem can be written as

$$\begin{aligned} \min \qquad & f(\mathbf{c}) = \mathbf{c}^T P^T P \mathbf{c} - (c_1 + 4c_2 + 13c_3 + 16c_4 + 5c_5) \\ \text{subject to} \qquad & \sum_{i=1}^{n} c_i = 1 \\ & c_i \geq 0, \text{ for } i = 1, \ldots, n \end{aligned}$$

Or in the standard form, it is

$$\begin{aligned} \min \qquad & f(\mathbf{c}) = \mathbf{c}^T P^T P \mathbf{c} - (c_1 + 4c_2 + 13c_3 + 16c_4 + 5c_5) \\ \text{subject to} \qquad & c_1 + c_2 + c_3 + c_4 + c_5 \geq 1 \\ & -c_1 - c_2 - c_3 - c_4 - c_5 \geq -1 \qquad\qquad c_i \geq 0, \text{ for } i = 1, \ldots, n \end{aligned}$$

This is a quadratic programming problem with

$$Q = 2P^t P = \begin{bmatrix} 2 & 4 & 6 & 8 & 4 \\ 4 & 8 & 12 & 16 & 8 \\ 6 & 12 & 26 & 24 & 16 \\ 8 & 16 & 24 & 32 & 16 \\ 4 & 8 & 16 & 16 & 10 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} -1 \\ -4 \\ -13 \\ -16 \\ -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

By using the Lemke's methods, we have the initial tableau

```
           z1    z2    z3    z4    z5    z6    z7    1
         ------------------------------------------------
w1  =  |    2     4     6     8     4    -1     1    -1
w2  =  |    4     8    12    16     8    -1     1    -4
w3  =  |    6    12    26    24    16    -1     1   -13
w4  =  |    8    16    24    32    16    -1     1   -16
w5  =  |    4     8    16    16    10    -1     1    -5
w6  =  |    1     1     1     1     1     0     0    -1
w7  =  |   -1    -1    -1    -1    -1     0     0     1
```

By adding the artificial variable z_0, we have

```
           z1    z2    z3    z4    z5    z6    z7    z0    1
         ------------------------------------------------------
w1  =  |    2     4     6     8     4    -1     1     1    -1
w2  =  |    4     8    12    16     8    -1     1     1    -4
w3  =  |    6    12    26    24    16    -1     1     1   -13
w4  =  |    8    16    24    32    16    -1     1     1   -16
w5  =  |    4     8    16    16    10    -1     1     1    -5
w6  =  |    1     1     1     1     1     0     0     1    -1
w7  =  |   -1    -1    -1    -1    -1     0     0     1     1
```

Following the Lemke's algorithm, exchange z_0 with w_4, then z_4 with w_3, then z_3 with w_6, then z_6 with w_1, and finally z_1 with z_0, we have the optimal tableau

	z0	z2	w6	w3	z5	w1	z7	w4	1
z6 = |	1.0000	0.0000	-0.0000	0.0000	-0.0000	-1.3333	1.0000	0.3333	4.0000
w2 = |	-0.0000	-0.0000	0.0000	-0.0000	0.0000	0.6667	0.0000	0.3333	2.0000
z4 = |	0.3333	-0.3333	-0.3333	-0.0833	-0.0000	-0.0278	0.0000	0.1111	0.3333
z1 = |	-1.3333	-0.6667	1.3333	-0.0417	-0.5000	0.0694	0.0000	-0.0278	0.4167
w5 = |	-0.0000	0.0000	0.0000	0.5000	0.0000	0.5000	0.0000	0.0000	2.0000
z3 = |	0.0000	0.0000	-0.0000	0.1250	-0.5000	-0.0417	0.0000	-0.0833	0.2500
w7 = |	2.0000	0.0000	-1.0000	0.0000	-0.0000	-0.0000	0.0000	-0.0000	0.0000

From the optimal tableau we see that the optimal solution is

$$\bar{\mathbf{c}} = (0.4167,\ 0,\ 0.25,\ 0.3333,\ 0)^t$$

and the minimum value of f is

$$min\, f = f(\bar{\mathbf{c}}) = -2.5$$

Hence the center of the circle is

$$x = 2.4999 \approx 2.5, \qquad y = 0.5$$

and the radius of the circle is

$$r = \sqrt{-f(\bar{\mathbf{c}})} = \sqrt{2.5} \approx 1.5911$$