Math 4553, Solution to Homework 4

1. To check whether an optimization problem is convex or not, we only need to check to two things: (1) Is the feasible region convex? (2) Is the objective function convex?

   (a) Clearly, the feasible region for this problem is convex since it is defined by linear inequality type constraints. To check the objective function, notice that

   $$\nabla^2 f = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

   It has two eigenvalues 1 and 3. Both eigenvalues are non-negative, therefore the objective function is convex. Combine the above, the optimization problem is convex.

   (b) Notice there is a nonlinear equation type constraints $x^2 + y^2 = 50$. Hence the feasible region can not be convex. The optimization problem is not convex.

2. This is a typical quadratic programming problem.

   (a) Since all constraints are of linear inequality type, clearly the feasible region is convex. The Hessian of the objective function is

   $$\nabla^2 f = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

   It has eigenvalues 2 and $\frac{1}{2}$, both are non-negative. therefore the objective function is convex. Combine the above, the optimization problem is convex.

   (b) The KKT conditions for quadratic programming problems have the form

   $$\begin{cases} 0 \le \mathbf{x} \perp Q\mathbf{x} - A^t\mathbf{u} + \mathbf{p} \ge 0 \\ 0 \le \mathbf{u} \perp A\mathbf{x} - \mathbf{b} \ge 0 \end{cases}$$

   for the given problem, we have

   $$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$
   $$A = \begin{pmatrix} 1, -2 \\ 1, -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -2 \\ -7 \end{pmatrix}.$$

   Therefore

   $$Q\mathbf{x} - A^t\mathbf{u} + \mathbf{p} = \begin{pmatrix} 2x_1 - u_1 - u_2 + 8 \\ \frac{1}{2}x_1 + 2u_1 + u_2 - 1 \end{pmatrix},$$
   $$A\mathbf{x} - \mathbf{b} = \begin{pmatrix} x_1 - x_2 + 2 \\ x_1 - x_2 + 7 \end{pmatrix}.$$

   Combine the above, the KKT conditions can be written as:

   $$\begin{cases} x_1(2x_1 - u_1 - u_2 + 8) = 0 \\ x_2(\frac{1}{2}x_1 + 2u_1 + u_2 - 1) = 0 \\ u_1(x_1 - 2x_2 + 2) = 0 \\ u_2(x_1 - x_2 + 7) = 0 \\ x_1, x_2, u_1, u_2, 2x_1 - u_1 - u_2 + 8, \frac{1}{2}x_1 + 2u_1 + u_2 - 1, x_1 - 2x_2 + 2, x_1 - x_2 + 7 \ge 0 \end{cases}$$

(c) If $x_1 = 0$, $x_2 = c$, $u_1 = \frac{1}{4}$ and $u_2 = 0$ satisfies the KKT conditions, by substituting them into the KKT conditions written in part (b), we have

$$\begin{cases} 0 \times (0 - \frac{1}{4} - 0 + 8) = 0 \\ c \times (\frac{1}{2}c + \frac{1}{2} + 0 - 1) = 0 \\ \frac{1}{4} \times (0 - 2c + 2) = 0 \\ 0 \times (0 - c + 7) = 0 \\ 0, c, \frac{1}{4}, 0, 0 - \frac{1}{4} - 0 + 8, \frac{1}{2}c + \frac{1}{2} + 0 - 1, 0 - 2c + 2, 0 - c + 7 \geq 0 \end{cases}$$

which implies

$$\begin{cases} c = 0 \text{ or } \frac{1}{2}c - \frac{1}{2} = 0 \\ -2c + 2 = 0 \\ c, \frac{1}{2}c - \frac{1}{2}, -2c + 2, -c + 7 \geq 0 \end{cases}$$

Clearly, the only value of c that satisfies all these conditions is $c = 1$.

(d) From part (a) and (c), we know that $x_1 = 0$, $x_2 = 1$ satisfies the KKT conditions for the convex quadratic programing problem. Hence it must be a global solution to this problem. And the minimum value is $f(0, 1) = -\frac{3}{4}$.