Math 4553, Solution to Homework 3

1. First, we need to form the linear programming problem. Denote e_{ij} to be the edge from node i to node j. Let x_{ij} be the flow on e_{ij}. If $x_{ij} = 1$, then the route goes through edge e_{ij}. If $x_{ij} = 0$, then the route does not go through e_{ij}. Altogether, we have 16 variables:

$$x_{12}, x_{13}, x_{14}, x_{25}, x_{26}, x_{27}, x_{35}, x_{37}, x_{38}, x_{48}, x_{56}, x_{57}, x_{58}, x_{69}, x_{79}, x_{89}$$

The optimization proble can be formed as

$$\begin{aligned}
\min \quad & f = 15x_{12} + 10x_{13} + 18x_{14} + 16x_{25} + 18x_{26} + 17x_{27} + 12x_{35} + 10x_{37} + 13x_{38} \\
& \quad + 10x_{48} + 15x_{56} + 16x_{57} + 14x_{58} + 16x_{69} + 23x_{79} + 12x_{89} \\
\text{subject to} \quad & x_{12} + x_{13} + x_{14} = 1 \\
& x_{25} + x_{26} + x_{27} - x_{12} = 0 \\
& x_{35} + x_{37} + x_{38} - x_{13} = 0 \\
& x_{48} - x_{14} = 0 \\
& x_{56} + x_{57} + x_{58} - x_{25} - x_{35} = 0 \\
& x_{69} - x_{26} - x_{56} = 0 \\
& x_{79} - x_{27} - x_{37} - x_{57} = 0 \\
& x_{89} - x_{38} - x_{48} - x_{58} = 0 \\
& -x_{69} - x_{79} - x_{89} = -1 \\
& x_{ij} \geq 0
\end{aligned}$$

we will solve the problem using the simplex method. The problem is not in the standard form, so we will use scheme 2 to deal with equation-type constraints. The initial tableau is

	x12	x13	x14	x25	x26	x27	x35	x37	x38	x48	x56	x57	x58	x69	x79	x89	1
y1 =	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	-1
y2 =	-1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0
y3 =	0	-1	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0
y4 =	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
y5 =	0	0	0	-1	0	0	-1	0	0	0	1	1	1	0	0	0	0
y6 =	0	0	0	0	-1	0	0	0	0	0	-1	0	0	1	0	0	0
y7 =	0	0	0	0	0	-1	0	-1	0	0	0	-1	0	0	1	0	0
y8 =	0	0	0	0	0	0	0	0	-1	-1	0	0	-1	0	0	1	0
y9 =	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	1
f =	15	10	18	16	18	17	12	10	13	10	15	16	14	16	23	12	0

Next, we exchange $y1$, $y2$, $y3$, $y4$, $y5$, $y6$, $y7$, $y8$ with $x12$, $x13$, , $x14$, $x25$, $x26$, $x27$, $x38$, $x69$, respectively. This shall be down in 8 consequent steps. Notice that in each step, the pivot entry should be non-zero in order to guaranttee that the Jordan exchange can be performed. After these 8 Jordan exchanges, we have the new tableau

	y1	y2	y3	y4	y5	y6	x35	x37	y7	x48	x56	x57	x58	y8	x79	x89	1
x12=	1	0	1	1	0	0	−1	−1	0	0	−0	−0	1	1	0	−1	1
x13=	0	0	−1	0	0	0	1	1	0	−1	−0	−0	−1	−1	0	1	0
x14=	0	0	0	−1	0	0	0	0	0	1	−0	−0	−0	−0	−0	−0	0
x25=	0	0	0	0	−1	0	−1	0	0	0	1	1	1	−0	−0	−0	0
x26=	1	1	1	1	1	0	−0	0	1	0	−1	0	0	1	−1	−1	1
x27=	0	0	0	0	0	0	−0	−1	−1	0	−0	−1	0	0	1	−0	0
x38=	0	0	0	0	0	0	−0	−0	0	−1	−0	−0	−1	−1	0	1	0
x69=	1	1	1	1	1	1	−0	−0	1	−0	−0	−0	−0	1	−1	−1	1
y9 =	−1	−1	−1	−1	−1	−1	0	0	−1	0	0	0	0	−1	0	0	0
f =	49	34	39	31	18	16	−9	−12	17	5	13	15	22	26	6	−14	49

The variables $y1$, $y2$, $y3$, $y4$, $y5$, $y6$, $y7$, $y8$ are now non-basic and will be set to be 0. Therefore, we can delete these columns. The tableau now becomes

	x35	x37	x48	x56	x57	x58	x79	x89	1
x12=	−1	−1	0	−0	−0	1	0	−1	1
x13=	1	1	−1	−0	−0	−1	0	1	0
x14=	0	0	1	−0	−0	−0	−0	−0	0
x25=	−1	0	0	1	1	1	−0	−0	0
x26=	−0	0	0	−1	0	0	−1	−1	1
x27=	−0	−1	0	−0	−1	0	1	−0	0
x38=	−0	−0	−1	−0	−0	−1	0	1	0
x69=	−0	−0	−0	−0	−0	−0	−1	−1	1
y9 =	0	0	0	0	0	0	0	0	0
f =	−9	−12	5	13	15	22	6	−14	49

Notice that $y9$ can not be exchanged with any non-basic variables. Indeed, the entire row corresponding to $y9$ is 0. This is because the constrains are linearly dependent. In this case, we can safely remove the zero row. And the remaining tableau is

	x35	x37	x48	x56	x57	x58	x79	x89	1
x12=	−1	−1	0	−0	−0	1	0	−1	1
x13=	1	1	−1	−0	−0	−1	0	1	0
x14=	0	0	1	−0	−0	−0	−0	−0	0
x25=	−1	0	0	1	1	1	−0	−0	0
x26=	−0	0	0	−1	0	0	−1	−1	1
x27=	−0	−1	0	−0	−1	0	1	−0	0
x38=	−0	−0	−1	−0	−0	−1	0	1	0
x69=	−0	−0	−0	−0	−0	−0	−1	−1	1
f =	−9	−12	5	13	15	22	6	−14	49

Now we have taken care of all equation-type constraints, and can start the normal two-phase simplex procedure. Notice the tableau is already feasible. There is no need to apply phase I. We can proceed to phase II directly. Pick $x89$ as the pivot column and $x12$ as the pivot row, perform the Jordan exchange, we have

```
         x35  x37  x48  x56  x57  x58  x79  x12  1
       --------------------------------------------
x89= |   -1   -1    0   -0   -0    1    0   -1    1
x13= |    0    0   -1   -0   -0    0    0   -1    1
x14= |    0    0    1   -0   -0   -0   -0    0    0
x25= |   -1    0    0    1    1    1   -0    0    0
x26= |    1    1    0   -1    0   -1   -1    1    0
x27= |   -0   -1    0   -0   -1    0    1    0    0
x38= |   -1   -1   -1   -0   -0    0    0   -1    1
x69= |    1    1   -0   -0   -0   -1   -1    1    0
       --------------------------------------------
f  = |    5    2    5   13   15    8    6   14   35
```

This is an optimal tableau. The optimal solution is

$$x_{13} = x_{38} = x_{89} = 1,$$

and all other x_{ij} are equal to 0. This means the shortest route should go through edges e_{13}, e_{38}, and then e_{89}. The distance is $f = 35$.

2. Notice that

$$A = \begin{bmatrix} 2 & 4 & 3 & 0 & -1 \\ 1 & -1 & 0 & 5 & 1 \\ 2 & 2 & -2 & 3 & 0 \end{bmatrix}, \qquad p = \begin{bmatrix} -2 \\ 5 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 17 \\ 12 \end{bmatrix}$$

and hence

$$A_B = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 1 \\ 2 & 3 & 0 \end{bmatrix}, \qquad A_N = \begin{bmatrix} 4 & 3 \\ -1 & 0 \\ 2 & -2 \end{bmatrix}$$

$$p_B = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad p_N = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

The current tableau can be calculated by the formula

	x_N	1
x_B	$-A_B^{-1}A_N$	$A_B^{-1}b$
f	$p_N^t - p_B^t A_B^{-1} A_N$	$p_B^t A_B^{-1} b$

It is not hard to compute that

$$A_B^{-1} = \begin{bmatrix} -3 & -3 & 5 \\ 2 & 2 & -3 \\ -7 & -6 & 10 \end{bmatrix}$$

By using the formula, we can compute that the current tableau is

	x_2	x_3	1
x_1	-1	19	3
x_4	0	-12	2
x_5	2	41	4
f	9	-5	0

This tableau is feasible but not optimal. One can pick the pivot column to be x_3 and the pivot row to be x_4.

3. The quadratic forms can be written as $\frac{1}{2}\mathbf{x}^t A\mathbf{x}$, where A is

(a) $A = \begin{bmatrix} 12 & -3 & 0 \\ -3 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ -3 & 0 & 13 & 0 \\ 2 & 1 & 0 & 14 \end{bmatrix}$