Math 4553, Exam II, Mar. 30, 2009

Name: ____________________

Score:

Each problem is worth 10 points. The total is 50 points.

1. Consider the unconstrained problem: minimizing $f(x_1, x_2) = x_1^2 + 2x_2^2 + 2x_1 + 3$. Given an initial point $\mathbf{x}^0 = (0, 1)$ and a direction $\mathbf{d}^0 = (-1, 0)$, show that the given direction is a descent direction at $\mathbf{x}^0$. Furthermore, compute the optimum step length using analytical line search, and write down the new point.

   Solution. Clearly
$$\nabla f = \begin{pmatrix} 2x_1 + 2 \\ 4x_2 \end{pmatrix} \quad \Rightarrow \quad \nabla f(\mathbf{x}^0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$
   Therefore
$$\nabla f(\mathbf{x}^0) \cdot \mathbf{d}^0 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -2 < 0.$$
   Hence $\mathbf{d}^0$ is a descent direction at $\mathbf{x}^0$.

   Then we have
$$\phi(\alpha) = f(\mathbf{x}^0 + \alpha\mathbf{d}^0) = f(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 0 \end{pmatrix}) = f(-\alpha, 1) = \alpha^2 - 2\alpha + 5.$$
   Notice that
$$\phi'(\alpha) = 2\alpha - 2, \qquad \phi''(\alpha) = 2.$$
   We can conclude that $\alpha = 1$ gives a local minimum of $\phi(\alpha)$. So the step size should be $\alpha = 1$, and the next point should be
$$\mathbf{x}^1 = \mathbf{x}^0 + \alpha\mathbf{d}^0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2. Write down the Lagrangian function of the following problem, and list all equations/inequalities in KKT(Karush-Kuhn-Tucker) conditions (you do NOT need to solve the system):

$$\begin{aligned} \text{minimizing} \quad & f(x,y,z) = x^2 + 9y^2 + z^2 \\ \text{subject to} \quad & xy \geq 1 \\ & x^2 + z^2 = 4 \\ & z \geq 0 \end{aligned}$$

Solution. The Lagrangian function is

$$L = (x^2 + 9y^2 + z^2) + u_1(1 - xy + s_1^2) + u_2(-z + s_2^2) + v(x^2 + z^2 - 4),$$

where $u_1,\ u_2 \geq 0$.

The KKT conditions are

$$\begin{aligned} & 2x - u_1 y + 2vx = 0 \\ & 18y - u_1 x = 0 \\ & 2z - u_2 + 2vz = 0 \\ & 1 - xy + s_1^2 = 0 \\ & -z + s_2^2 = 0 \\ & x^2 + y^2 - 4 = 0 \\ & u_i s_i = 0, \quad \text{for } i = 1,2 \\ & s_1^2 \geq 0, \quad \text{for } i = 1,2 \\ & u_i \geq 0, \quad \text{for } i = 1,2 \end{aligned}$$

3. Rewrite the following problem into a standard LP (linear programming) form, and find all basic feasible solutions.

$$\begin{aligned} \text{minimizing} \quad & f(x_1, x_2) = -x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 = 9 \\ & x_1 + x_2 \geq -3 \\ & x_1 \geq 1 \end{aligned}$$

Solution. Define new variables

$$y_1 = x_1 - 1 \quad \text{and} \quad x_2 = y_2 - y_3,$$

where $y_i \geq 0$ for $i = 1, 2, 3$. Then, substitute $x_1 = y_1 + 1$ and $x_2 = y_2 - y_3$ into the original problem, we have

$$\begin{aligned} \text{minimize} \quad & f = -(y_1 + 1) - (y_2 - y_3) \\ \text{with} \quad & y_1 + 1 + 2(y_2 - y_3) = 9 \\ & (y_1 + 1) + (y_2 - y_3) \geq -3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \text{minimize} \quad & \tilde{f} = -y_1 - y_2 + y_3 \\ \text{with} \quad & y_1 + 2y_2 - 2y_3 = 8 \\ & -y_1 - y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Add a slack variable y_4 to the less than type constraint, we get the standard form

$$\begin{aligned} \text{minimize} \quad & \tilde{f} = (-1, -1, 1, 0)\mathbf{y} \\ \text{with} \quad & \begin{pmatrix} 1 & 2 & -2 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix} \mathbf{y} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & \mathbf{y} \geq 0 \end{aligned}$$

where $\mathbf{y} = (y_1, y_2, y_3, y_4)^t$.

There are $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{4!}{2!2!} = 6$ combinations of basic variables:

Basic	Solution	
$[y_1, y_2]$	(-16, 12, 0,0)	-
$[y_1, y_3]$	(-16, 0, -12,0)	-
$[y_1, y_4]$	(8, 0, 0,12)	BFS
$[y_2, y_3]$	-	-
$[y_2, y_4]$	(0,4,0,8)	BFS
$[y_3, y_4]$	(0,0,-4,8)	-

4. By adding artificial variables and then eliminating them, find a basic feasible solution to the problem:

$$\begin{aligned} \text{minimizing} \quad & f(x_1, x_2, x_3) = -3x_1 - x_3 \\ \text{subject to} \quad & 2x_2 + 2x_3 = 15 \\ & x_1 - 2x_2 + x_3 = 6 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Note: You only need to complete phase I of the two-phase method.

Solution. For each equation type constraint, we add one artificial variable. The artificial function is defined to be $g = a_1 + a_2$.

Initial Tableau

	x_1	x_2	x_3	a_1	a_2	rhs
$Art.$	0	0	0	1	1	g
a_1	0	2	2	1	0	15
a_2	1	-2	1	0	1	6

First, we need to make artificial function depend only on non-basic variables. This is done by performing Row1 - Row2 - Row3:

	x_1	x_2	x_3	a_1	a_2	rhs
$Art.$	-1	0	-3	0	0	$g-21$
a_1	0	2	2	1	0	15
a_2	1	-2	$\boxed{1}$	0	1	6

Next, we perform the pivoting steps. The first pivot column is column 3 and the pivot row is row 3. It has been denoted by the framed number in the above tableau. After the pivoting, x_3 replaces a_2 in the set of basic variables, and we have

	x_1	x_2	x_3	a_1	a_2	rhs
$Art.$	2	-6	0	0	3	$g-3$
a_1	-2	$\boxed{6}$	0	1	-2	3
x_3	1	-2	1	0	1	6

The next pivot column and row are indicated again by the framed number in the above tableau. After that, we have

	x_1	x_2	x_3	a_1	a_2	rhs
$Art.$	0	0	0	1	1	g
x_2	$-\frac{1}{3}$	1	0	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$
x_3	$\frac{1}{3}$	0	1	$\frac{1}{3}$	$\frac{1}{3}$	7

Notice that a_1 and a_2 are now both non-basic. We can stop here and the BFS is:

$$x_1 = 0, \quad x_2 = \frac{1}{2}, \quad x_3 = 7.$$

5. Solve following problem using the simplex method:

$$\begin{aligned} \text{maximizing} \quad & f(x_1, x_2, x_3) = -3x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & 4x_1 + 5x_2 - 2x_3 \leq 22 \\ & x_1 - 2x_2 + x_3 \leq 30 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

In each step of the simplex algorithm, also compute the current basic feasible solution and the value of function f at this point.

Solution. First, we need to change the objective function into

$$\text{maximizing} \quad \tilde{f} = 3x_1 - 2x_2 - 4x_3.$$

For each less than type constraint, a slack variable will be added to transform them into the standard form.

Initial Tableau: BFS = (0, 0, 0, 22, 30), $\tilde{f}$ = 0, f = 0

	x_1	x_2	x_3	x_4	x_5	rhs
$Obj.$	3	-2	-4	0	0	$\tilde{f}$
x_4	4	5	-2	1	0	22
x_5	1	-2	$\boxed{1}$	0	1	30

Second tableau: BFS = (0, 0, 30, 82, 0), $\tilde{f}$ = -120, f = 120

	x_1	x_2	x_3	x_4	x_5	rhs
$Obj.$	7	-10	0	0	4	$\tilde{f} + 120$
x_4	6	$\boxed{1}$	0	1	2	82
x_3	1	-2	1	0	1	30

Third tableau: BFS = (0, 82, 194, 0, 0), $\tilde{f}$ = -940, f = 940

	x_1	x_2	x_3	x_4	x_5	rhs
$Obj.$	67	0	0	10	24	$\tilde{f} + 940$
x_2	6	1	0	1	2	82
x_3	13	0	1	2	5	194

The final solution is:

$$x_1 = 0, \quad x_2 = 82, \quad x_3 = 194$$

and

$$f(0,\ 82,\ 194) = 940.$$