Math 4553, Exam I, Feb. 13, 2009

Name: ____________________

Score:

Each problem is worth 10 points. The total is 50 points.

1. A furniture manufacturer is trying to maximize the weekly revenue of a factory. Four products, their requirements of wood and labor, and the profit of each product are listed in the table below. Suppose that 5000 units of wood and 1500 units of labor are available each week. Formulate the optimization problem that determines how many units of each product the factory should produce to maximize its profit.

Product	Wood	Labor	Profit
Bookshelf	10	2	100
Coffee table	12	4	150
Cabinet	25	8	200
Deluxe cabinet	20	12	400

Solution

(a) Set the optimization variables as

$$\begin{aligned} x_1 &= \text{number of bookshelves made per week} \\ x_2 &= \text{number of coffee tables made per week} \\ x_3 &= \text{number of cabinets made per week} \\ x_4 &= \text{number of deluxe cabinets made per week} \end{aligned}$$

(b) The objective is to

$$\text{maximize } f(x_1, x_2, x_3, x_4) = 100x_1 + 150x_2 + 200x_3 + 400x_4$$

(c) Subject to the constraints

$$\begin{array}{ll} \text{Wood:} & 10x_1 + 12x_2 + 25x_3 + 20x_4 \le 5000 \\ \text{Labor:} & 2x_1 + 4x_2 + 8x_3 + 12x_4 \le 1500 \\ \text{Other:} & x_1,\ x_2,\ x_3,\ x_4 \ge 0, \text{ and they must be integers} \end{array}$$

2. A manufacturer wants to design a cylindrical can to hold at least 400ml of liquid. Fabrication, handling, aesthetics, and shipping considerations impose the following restrictions on the can: the diameter should be no more than 8cm and no less than 3.5cm, whereas the height should be no more than 18cm and no less than 8cm. The cost of making a can is proportional to the surface area of the sheet metal used (including top and bottom lids of the can). Formulate an optimization problem to help the manufacturer minimize the cost while subject to given constraints.

   1ml = 1cm^3

   Solution

   (a) Set the optimization variables as

   $$\begin{aligned} r &= \text{the radius of the can} \\ h &= \text{the height of the can} \end{aligned}$$

   (Note: you can also set the variables to be the diameter and the height.)

   (b) The objective is to minimize the cost of making the can, which is equivalent to minimize the surface area of the can in this problem:

   $$\text{minimize } f(r,h) = (2\pi r)\,h + 2(\pi r^2)$$

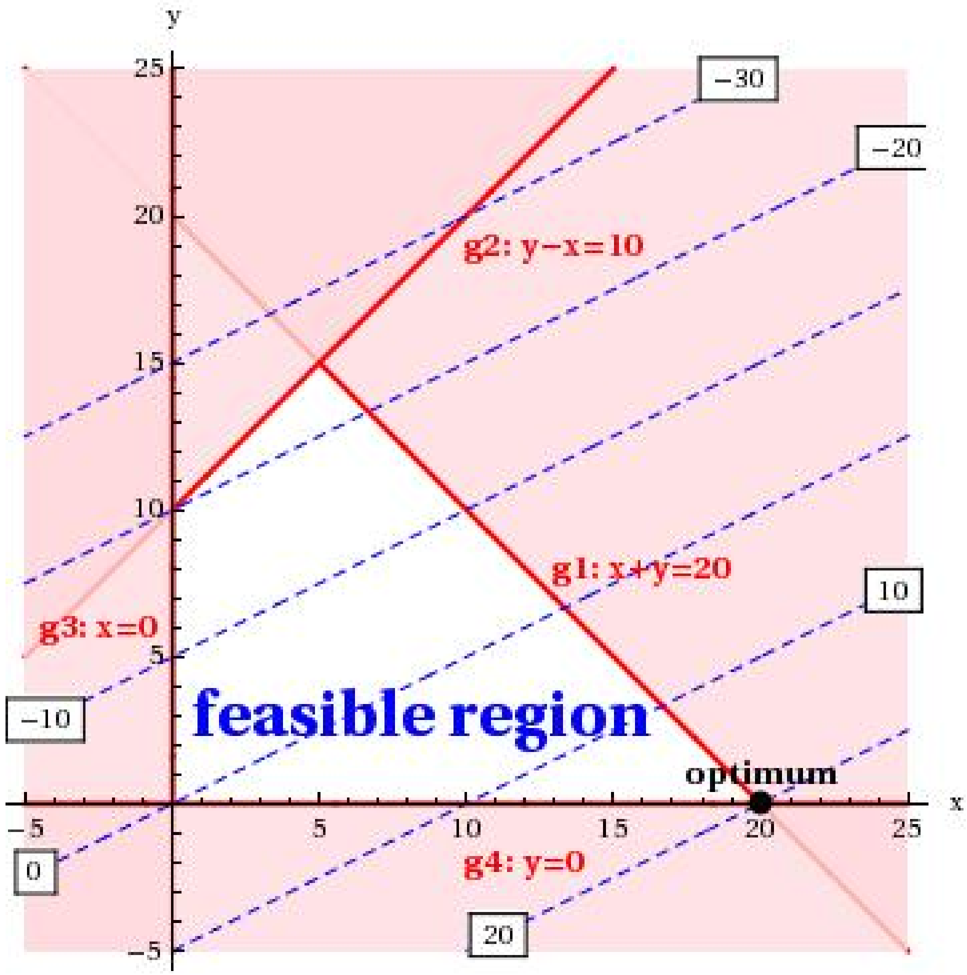
   (c) Subject to the constraints

   $$\begin{aligned} &(\pi r^2)\,h \geq 400cm^3 \\ &3.5cm \leq 2r \leq 8cm \\ &8cm \leq h \leq 18cm \\ &r,\ h \geq 0 \end{aligned}$$

3. Solve the following problem using the graphical optimization:

$$\begin{array}{ll} \text{Maximize} & f(x,y) = x - 2y \\ \text{Subject to} & x + y \le 20 \\ & y - x \le 10 \\ & x \ge 0,\ y \ge 0 \end{array}$$

solution



The maximum is achieved at point $(20,0)$ and $f(20,0) = 20$.

4. Determine the region where the following function is convex:

$$f(x, y) = x^3 - 3xy + 4y^2$$

Solution First, we compute the Hessian of f:

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 8 \end{pmatrix}$$

By the principle minors test, we have

$$\begin{cases} 6x \geq 0 \\ (6x) \times (8) - (-3) \times (-3) = 48x - 9 \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \geq \frac{9}{48} \end{cases}$$

Combine the above, we know that f is convex when $x \geq \frac{9}{48}$.

5. Find all stationary points of the function $f(x,y) = 3x - x^3 - 2y^2 + y^4$, and determine whether they are local minimums, local maximums or saddle points.

Solution First, we compute the stationary points:

$$\nabla f = \begin{pmatrix} 3 - 3x^2 \\ -4y + 4y^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{cases} 3 - 3x^2 = 0 \\ -4y + 4y^3 = 0 \end{cases}$$

Notice that

$$\begin{aligned} &3 - 3x^2 = 3(1 - x^2) = 0 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = \pm 1 \\ &-4y + 4y^3 = 4y(y^2 - 1) = 0 \;\Rightarrow\; y = 0 \text{ or } y^2 = 1 \;\Rightarrow\; y = 0, \pm 1 \end{aligned}$$

Therefore, there are 6 stationary points: $(1,0)$, $(1,-1)$, $(1,1)$, $(-1,0)$, $(-1,-1)$, $(-1,1)$.

Next, we compute the Hessian of f:

$$\nabla^2 f = \begin{pmatrix} -6x & 0 \\ 0 & -4 + 12y^2 \end{pmatrix}$$

The eigenvalues of $\nabla^2 f$ are $-6x$ and $-4 + 12y^2$, since it is a diagonal matrix. By checking the eigenvalues of $\nabla^2 f$ at all stationary points, we have

point	eigenvalues	$\nabla^2 f$	point type
(1, 0)	-6, -4	negative definite	local maximum
(1,-1)	-6, 8	indefinite	saddle point
(1, 1)	-6, 8	indefinite	saddle point
(-1, 0)	6, -4	indefinite	saddle point
(-1,-1)	6, 8	positive definite	local minimum
(-1, 1)	6, 8	positive definite	local minimum