Solution to homework problem # 4.107 and 7.1

Let us first review some basic concepts.

1. Derive the dual problem from the Lagrangian duality. It works for convex problems, including all linear programming problems. Given a convex optimization problem

$$\begin{array}{ll} \text{Minimize} & f(\mathbf{x}) \\ \text{subject to} & \begin{pmatrix} g_i(\mathbf{x}) \le 0, & \text{for } i = 1, \dots, m_1 \\ h_i(\mathbf{x}) = 0, & \text{for } i = 1, \dots, m_2 \end{pmatrix} \end{array}$$

   let us first write down its Lagrangian function

$$L(\mathbf{x}, \mathbf{u}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{m_1} u_i g_i(\mathbf{x}) + \sum_{i=1}^{m_2} v_i h_i(\mathbf{x}),$$
$$\textbf{where } u_i \ge 0, \quad \text{for } i = 1, \dots, m_1,$$

   and v_i are free variables.

   Then we have

$$\begin{array}{lcl} \text{primal problem} & \Longleftrightarrow & \min_{\mathbf{x}} \left(\max_{\mathbf{u} \ge 0,\, \mathbf{v}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}) \right) \\ \text{dual problem} & \Longleftrightarrow & \max_{\mathbf{u} \ge 0,\, \mathbf{v}} \left(\min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}) \right) \end{array}$$

   To derive the dual problem, we need to compute $\min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v})$, which can be calculated by using $\nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}) = \mathbf{0}$. Indeed, $\nabla_{\mathbf{x}} L = \mathbf{0}$, together with $\mathbf{u} \ge 0$, gives the constraints of the dual problem. Suppose $g(\mathbf{u}, \mathbf{v}) = \min_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v})$, then the dual problem is

$$\begin{array}{ll} \text{Maximize} & g(\mathbf{u}, \mathbf{v}) \\ \text{subject to} & \begin{pmatrix} \nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{u}, \mathbf{v}) = \mathbf{0} \\ \mathbf{u} \ge 0 \end{pmatrix} \end{array}$$

   (Do not forget $\mathbf{u} \ge 0$, and DO NOT add $\mathbf{v} \ge 0$.)

2. For linear programming problems, we have some easy-to-use formulas. However, one need to be careful on choosing the correct formula to use. In the following table, the first primal problem is written in its canonical form, and the second is in its standard form.

primal problem	dual problem
Minimize $f = \mathbf{c}^t\mathbf{x}$ subject to $\begin{pmatrix} A\mathbf{x} \ge \mathbf{b} \\ \mathbf{x} \ge 0 \end{pmatrix}$	Maximize $g = \mathbf{b}^t\mathbf{y}$ subject to $\begin{pmatrix} A^t\mathbf{y} + \mathbf{u} = \mathbf{c} \\ \mathbf{y} \ge 0,\ \mathbf{u} \ge 0 \end{pmatrix}$ or Maximize $g = \mathbf{b}^t\mathbf{y}$ subject to $\begin{pmatrix} A^t\mathbf{y} \le \mathbf{c} \\ \mathbf{y} \ge 0 \end{pmatrix}$
Minimize $f = \mathbf{c}^t\mathbf{x}$ subject to $\begin{pmatrix} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \ge 0 \end{pmatrix}$	Maximize $g = \mathbf{b}^t\mathbf{y}$ subject to $\begin{pmatrix} A^t\mathbf{y} + \mathbf{u} = \mathbf{c} \\ \mathbf{u} \ge 0 \end{pmatrix}$ or Maximize $g = \mathbf{b}^t\mathbf{y}$ subject to $A^t\mathbf{y} \le \mathbf{c}$

   (The formula given in textbook uses $\mathbf{v}$ instead of $\mathbf{y}$. This $\mathbf{v}$ is different from the $\mathbf{v}$ in Lagrangian duality forms.)

3. The Lagrangian duality in item 1 and the formulas in item 2 should give exactly the same dual problem. Sometime they may have differently-looking forms. However, by eliminating slack variables or using a change of variable, these forms can be shown to be equivalent.

Solution for # 4.107. The primal problem is

$$\begin{array}{ll} \text{Minimize} & f = -x - 3y \\ \text{subject to} & \begin{pmatrix} x + y = 6 \\ -x + y \le 4 \end{pmatrix} \end{array}$$

The Lagrangian function can be written as

$$L(x, y, u, v) = (-x - 3y) + u(-x + y - 4) + v(x + y - 6), \quad \textbf{where } u \ge 0.$$

Notice that v should be a free variable. Then the dual problem is equivalent to the following saddle point problem:

$$\max_{u \ge 0,\, v} \left(\min_{x,y} L(x, y, u, v) \right).$$

To compute $\min_{x,y} L(x, y, u, v)$, let us take

$$\begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{cases} -1 - u + v = 0 \\ -3 + u + v = 0 \end{cases},$$

and when u, v satisfy these two equations, we have

$$\min_{x,y} L(x, y, u, v) = (-1 - u + v)x + (-3 + u + v)y - 4u - 6v = 0x + 0y - 4u - 6v.$$

Substitute this into $\max_{u \ge 0,\, v} (\min_{x,y} L(x, y, u, v))$, the dual problem can be written as

$$\boxed{\begin{array}{ll} \text{Maximize} & g = -4u - 6v \\ \text{subject to} & \begin{pmatrix} -1 - u + v = 0 \\ -3 + u + v = 0 \\ u \ge 0 \end{pmatrix} \end{array}}$$

(Notice that $u \ge 0$ is required by the Lagrangian function, and v should be a free variable.)

It is not hard to find that the primal problem has $\min f = -16$ at $(x, y) = (1, 5)$, and the dual problem has $\max g = -16$ at $(u, v) = (1, 2)$. The details are skipped.

Alternative solution for # 4.107. You can also solve the problem as follows. The primal problem is equivalent to

$$\begin{array}{ll} \text{Minimize} & f = -x - 3y \\ \text{subject to} & \begin{pmatrix} x + y \le 6 \\ -x - y \le -6 \\ -x + y \le 4 \end{pmatrix} \end{array}$$

Then the Lagrangian function becomes

$$L(x, y, u_1, u_2, u_3) = (-x - 3y) + u_1(x + y - 6) + u_2(-x - y + 6) + u_3(-x + y - 4)$$
$$\textbf{where } u_1,\ u_2,\ u_3 \ge 0.$$

Then $\min_{x,y} L(x, y, u_1, u_2, u_3)$ is taken at the point where

$$\begin{pmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{cases} -1 + u_1 - u_2 - u_3 = 0 \\ -3 + u_1 - u_2 + u_3 = 0 \end{cases},$$

and its values is

$$\min_{x,y} L(x, y, u_1, u_2, u_3) = -6u_1 + 6u_2 - 4u_3.$$

Hence the dual problem can also be written as

$$\boxed{\begin{array}{ll} \text{Maximize} & g = -6u_1 + 6u_2 - 4u_3 \\ \text{subject to} & \begin{pmatrix} -1 + u_1 - u_2 - u_3 = 0 \\ -3 + u_1 - u_2 + u_3 = 0 \\ u_1,\ u_2,\ u_3 \ge 0 \end{pmatrix} \end{array}}$$

(Notice that by setting $u = u_3$, which should satisfy $u \ge 0$, and $v = u_1 - u_2$, which should be a free variable, we get exactly the same form as in the previous solution.)

Solution for # 7.1. The primal problem is

$$\text{Maximize} \quad z = x_1 + 3x_2$$
$$\text{subject to} \quad \begin{pmatrix} x_1 + 4x_2 \le 10 \\ x_1 + 2x_2 \le 10 \\ x_1,\, x_2 \ge 0 \end{pmatrix}$$

This problem can be rewritten into a canonical form:

$$\begin{aligned} &\text{Minimize} && f = -x_1 - 3x_2 \\ &\text{subject to} && \begin{pmatrix} -1 & -4 \\ -1 & -2 \end{pmatrix} \mathbf{x} \ge \begin{pmatrix} -10 \\ -10 \end{pmatrix} \\ & && x_1,\, x_2 \ge 0 \end{aligned}$$

Hence by the formula for the canonical form (on page 1 of this document), we have the dual problem:

$$\boxed{\begin{aligned} &\text{Maximize} && g = -10y_1 - 10y_2 \\ &\text{subject to} && \begin{pmatrix} -y_1 - y_2 + u_1 = -1 \\ -4y_1 - 2y_2 + u_2 = -3 \\ \mathbf{y} \ge 0,\, \mathbf{u} \ge 0 \end{pmatrix} \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} &\text{Maximize} && g = -10y_1 - 10y_2 \\ &\text{subject to} && \begin{pmatrix} -y_1 - y_2 \le -1 \\ -4y_1 - 2y_2 \le -3 \\ \mathbf{y} \ge 0 \end{pmatrix} \end{aligned}}$$

If you prefer to first write the primal problem into its standard form, that's fine. The standard form for the primal problem is

$$\begin{aligned} &\text{Minimize} && f = -x_1 - 3x_2 + 0x_3 + 0x_4 \\ &\text{subject to} && \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \\ & && x_1,\, x_2,\, x_3,\, x_4 \ge 0 \end{aligned}$$

Use the formula for the standard form (on page 1 of this document), we have the dual problem:

$$\boxed{\begin{aligned} &\text{Maximize} && g = 10y_1 + 10y_2 \\ &\text{subject to} && \begin{pmatrix} y_1 + y_2 + u_1 = -1 \\ 4y_1 + 2y_2 + u_2 = -3 \\ y_1 + u_3 = 0 \\ y_2 + u_4 = 0 \\ \mathbf{u} \ge 0 \end{pmatrix} \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} &\text{Maximize} && g = 10y_1 + 10y_2 \\ &\text{subject to} && \begin{pmatrix} y_1 + y_2 \le -1 \\ 4y_1 + 2y_2 \le -3 \\ y_1 \le 0 \\ y_2 \le 0 \end{pmatrix} \end{aligned}}$$

It is not hard to see that all above four different forms (in 4 boxes) for the dual problem are equivalent. For example, take the two forms on the right. By changing the sign of variables y_1 and y_2, we can see that they are exactly the same.

Finally, one can also use Lagrangian duality to find the dual for this problem. There are two possibilities.

1. Use the Lagrangian function for the canonical form, which should be

$$L(\mathbf{x}, \mathbf{u}) = (-x_1 - 3x_2) + u_1(x_1 + 4x_2 - 10) + u_2(x_1 + 2x_2 - 10) + u_3(-x_1) + u_4(-x_2)$$
$$\textbf{where } u_1,\, u_2,\, u_3,\, u_4 \ge 0.$$

   It is important not to forget constraints $-x_1 \le 0$ and $-x_2 \le 0$ in the Lagrangian function. Then using the formula for the Lagrangian duality (on page 1 of this document), we have the dual problem

$$\boxed{\begin{aligned} &\text{Maximize} && g = -10u_1 - 10u_2 \\ &\text{subject to} && \begin{pmatrix} -1 + u_1 + u_2 - u_3 = 0 \\ -3 + 4u_1 + 2u_2 - u_4 = 0 \\ u_1,\, u_2,\, u_3,\, u_4 \ge 0 \end{pmatrix} \end{aligned}}$$

2. Use the Lagrangian function for the standard form, which should be

$$
\begin{aligned}
L(\mathbf{x},\mathbf{u},\mathbf{v}) = &(-x_1 - 3x_2) + v_1(x_1 + 4x_2 + x_3 - 10) + v_2(x_1 + 2x_2 + x_4 - 10) \\
&+ u_1(-x_1) + u_2(-x_2) + u_3(-x_3) + u_4(-x_4) \\
&\qquad\qquad\qquad\qquad \textbf{where } u_1,\ u_2,\ u_3,\ u_4 \geq 0.
\end{aligned}
$$

Again, it is important not to forget constraints $-x_i \leq 0$, for $i = 1, 2, 3, 4$, in the Lagrangian function. Also, notice that v_1, v_2 are free variables, since they corresponds to "equation" type constraints of the standard form. Then using the formula for the Lagrangian duality, the dual problem is

$$
\boxed{
\begin{array}{ll}
\text{Maximize} & g = -10v_1 - 10v_2 \\
\text{subject to} & \left(\begin{array}{c}
-1 + v_1 + v_2 - u_1 = 0 \\
-3 + 4v_1 + 2v_2 - u_2 = 0 \\
v_1 - u_3 = 0 \\
v_2 - u_4 = 0 \\
u_1,\ u_2,\ u_3,\ u_4 \geq 0
\end{array}\right)
\end{array}
}
$$

Altogether, for problem # 7.1, we already have 6 different forms for the dual problem. All of them are equivalent. (this is left as an exercise for you.)

It can also be shown that the minimum of the primal problem is -10, and the maximum of the dual problem (no matter in which form) is -10.