Math 4513, Homework 5, Due on 12/5/2014

1. (5 points) Use Taylor's method of order 2 to approximate the solution for the following problem:
$$y' = 1 + (t-y)^2, \qquad y(2) = 1$$
Use step size $h = 0.5$ to estimate the value of $y(3)$.

2. (5 points) Find the computational cost of backward substitution, i.e., let A be an $n \times n$ upper-triangular matrix and $\mathbf{b}$ be an n-dimensional vector, one can solve $A\mathbf{x} = \mathbf{b}$ by computing x_n, x_{n-1}, x_{n-2}, $\cdots$, x_1 one-by-one.

3. (5 points) Use Gaussian elimination with scaled partial pivoting to solve the following problem:
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 12x_2 - x_3 &= 4 \\ 2x_1 + x_2 + x_3 &= 5 \end{aligned}$$
Write down the augemented matrix at each iteration step.

4. (5 points) Consider a linear system $A\mathbf{x} = \mathbf{b}$. Due to round-off errors, a numerical method implemented in the computer can only give us an approximation $\tilde{\mathbf{x}}$ to the exact solution $\mathbf{x}$. How good the approximation is can be measured by the Euclidean distance between vectors $\tilde{\mathbf{x}}$ and $\mathbf{x}$, if one knows the exact solution. This Euclidean distance is called the error.

   In practice, numerical method does not always give good approximations, i.e., the error can be large. It depends on the condition number of the coefficient matrix A. Perform the following numerical experiments and describe what you observe.

   (a) Define an $n^2 \times n^2$ matrix by `A = full(gallery('poisson',n))`, which is the coefficient matrix from discretizing the Poisson problem. The `gallery` command generates a sparse matrix, and we use the `full` command to convert it into a full matrix. Let $\mathbf{x}$ be an n^2-dimensional vector consisting of all 1s, and use it to compute $\mathbf{b} = A\mathbf{x}$. Next, use the command

   ```
   tildex = linsolve(A,b);
   ```

   to compute an approximation $\tilde{\mathbf{x}}$. The matlab command `linsolve` solves a linear system using the LU factorization (Gaussian elimination) with partial pivoting. Now we have the exact solution $\mathbf{x}$, which contains all 1s, and a numerical approximation $\tilde{\mathbf{x}}$. Compute the Euclidean distance between them using the command

   ```
   norm(x-tildex)
   ```

   Repeat this for $n = 2, 3, \cdots, 8$ and report the error.

   (b) Set matrix A to be an $n^2 \times n^2$ Hilbert matrix by using `A = hilb(n^2);` and repeat the above process. Report the error for $n = 2, 3, \cdots, 8$.

(You will observe that the error of the Hilbert matrix becomes very large. Fortunately, such a bad coefficient matrix is very rare in practice. The LU method with partial pivoting works well in most cases.)