Lagrange interpolation

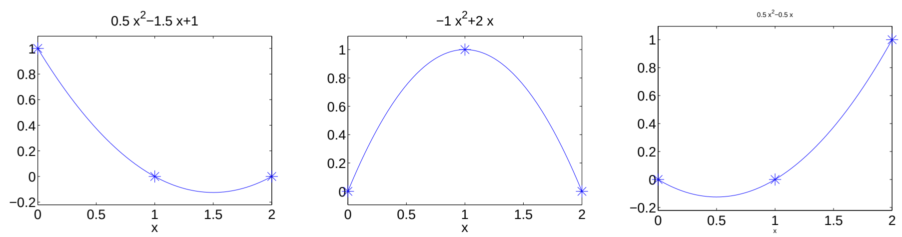
1. **An example with $n = 2$.** It is easy to check that quadratic polynomials

$$L_{2,0}(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1 \qquad \text{goes through 3 points } (0,1),\ (1,0),\ (2,0),$$

$$L_{2,1}(x) = -x^2 + 2x \qquad \text{goes through 3 points } (0,0),\ (1,1),\ (2,0),$$

$$L_{2,2}(x) = \frac{1}{2}x^2 - \frac{1}{2}x \qquad \text{goes through 3 points } (0,0),\ (1,0),\ (2,1).$$

Their graphs look like the following:



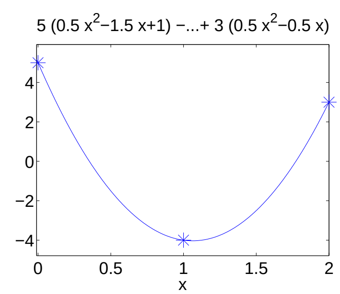
Now, if we would like to find a quadratic polynomial that goes throught points

$$(0,5),\ (1,-3),\ (2,4),$$

all we need to do is to compute the linear combination

$$5L_{2,0}(x) - 3L_{2,1}(x) + 4L_{2,2}(x) = 5\left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right) - 3\left(-x^2 + 2x\right) + 4\left(\frac{1}{2}x^2 - \frac{1}{2}x\right)$$

It is not hard to check the above quadratic polynomial goes through $(0,5),\ (1,-3),\ (2,4)$. Its graph is given below:



In general,

$$f_0 L_{2,0}(x) + f_1 L_{2,1}(x) + f_2 L_{2,2}(x) \qquad \text{goes through 3 points } (0, f_0),\ (1, f_1),\ (2, f_2).$$

2. **An example with $n = 3$.** Now let's look at the case of cubic polynomials. Clearly,

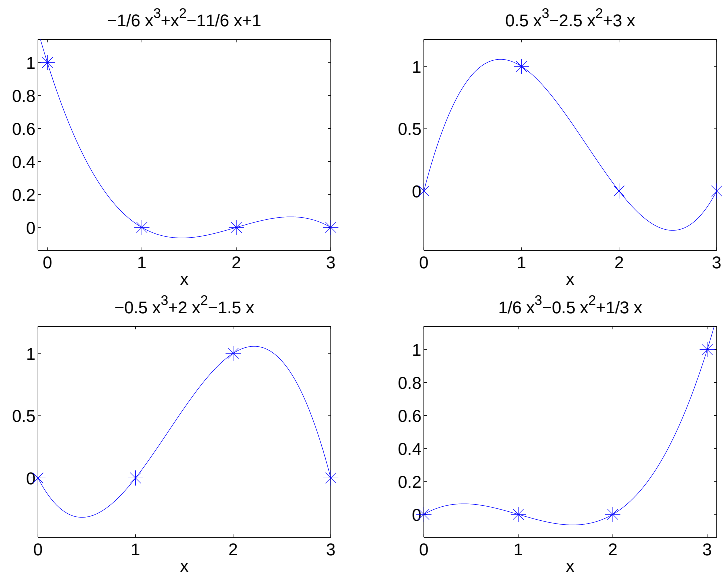
$$L_{3,0}(x) = -\frac{1}{6}x^3 + x^2 - \frac{11}{6}x + 1 \qquad \text{goes through 4 points } (0,1),\ (1,0),\ (2,0),\ (3,0),$$

$$L_{3,1}(x) = \frac{1}{2}x^3 - \frac{5}{2}x^2 + 3x \qquad \text{goes through 4 points } (0,0),\ (1,1),\ (2,0),\ (3,0),$$

$$L_{3,2}(x) = -\frac{1}{2}x^3 + 2x^2 - \frac{3}{2}x \qquad \text{goes through 4 points } (0,0),\ (1,0),\ (2,1),\ (3,0),$$

$$L_{3,3}(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x \qquad \text{goes through 4 points } (0,0),\ (1,0),\ (2,0),\ (3,1),$$

Their graphs look like the following:



and the linear combination

$f_0 L_{3,0}(x) + f_1 L_{3,1}(x) + f_2 L_{3,2}(x) + f_3 L_{3,3}$ goes through 4 points $(0, f_0),\ (1, f_1),\ (2, f_2),\ (3, f_3)$.