Math 4513, Midterm Exam, 10/21/2011

1. (20 points) Find the rate of convergence in $O(h^n)$ for

$$\lim_{h \to 0} \frac{1 - e^h}{h} = -1.$$

2. (20 points) Given $g(x) = \frac{1}{2}x + \frac{1}{x}$, consider the fixed point $p = g(p)$ in the interval $[1, 2]$. It has a unique fixed point $p = \sqrt{2} \approx 1.41421356$ in $[1, 2]$.

   (a) Use the fixed point theorem to prove that the iteration

   $$p_n = \frac{1}{2}p_{n-1} + \frac{1}{p_{n-1}}$$

   with initial value $p_0 = 2$ converges to $\sqrt{2}$.

   (b) Compute p_1, p_2 and p_3.

3. (20 points) Starting with $p_0 = 1.5$, find the approximation p_2 to the root of the equation $2x^2 - x - 2 = 0$ using the Newton's method.

4. (20 points) Use the divided differences to compute the Hermite interpolation for

$$\begin{aligned} x_0 &= -1, & x_1 &= 0, \\ f(x_0) &= 1, & f(x_1) &= 0, \\ f'(x_0) &= 1, & f'(x_1) &= 1, \end{aligned}$$

5. (20 points) Let $f(x) = 2x^2 - 3x + 1$. Define $x_0 = 1$ and $M = f'(x_0)$. Consider the backward difference $N_1(h) = \frac{f(x_0) - f(x_0 - h)}{h}$.

   (a) Compute the Taylor's expansion of $f(x_0 - h)$ at x_0, and then show that $M - N_1(h) = O(h)$.

   (b) Compute $N_1(1)$, $N_1(\frac{1}{2})$, $N_1(\frac{1}{4})$. Use these data and the Richardson's extrapolation to compute $N_3(1)$.