Solution the the Midterm Exam

1. By the Taylor expansion

$$\frac{1-e^h}{h} = \frac{1-(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots)}{h} = -1-\frac{h}{2!}-\frac{h^2}{3!}-\cdots$$

Hence it converges to -1 with order $O(h)$

2. (a) Notice that ① $g(x) = \frac{x}{2}+\frac{1}{x}$ is continuous in $[1,2]$

② $\min_{1\le x\le 2} g(x) = \sqrt{2}$ $\left(\text{by taking } g'(x)=0 \Rightarrow x=\sqrt{2}\right)$

$\max_{1\le x\le 2} g(x) = \frac{3}{2}$ (at $x=1$ and $x=2$)

$\Rightarrow \quad g(x) \in [\sqrt{2}, \frac{3}{2}] \subset [1,2]$

③ $g'(x) = \frac{1}{2}-\frac{1}{x^2}$ exists ~~for~~ in $[1,2]$

and $|g'(x)| \le \frac{1}{2} < 1$ for all $1\le x\le 2$

Then, by the fixed point theorem, the fixed point iteration converges to the unique fixed point, which is $\sqrt{2}$

(b) $p_0 = 2$

$p_1 = g(p_0) = \frac{2}{2}+\frac{1}{2} = \frac{3}{2}$

$p_2 = g(p_1) = \frac{3}{4}+\frac{2}{3} = \frac{17}{12}$

$p_3 = g(p_2) = \frac{17}{24}+\frac{12}{17} \approx \cdots$

3. Clearly, $f(x) = 2x^2 - x - 2$ and $f'(x) = 4x - 1$

By the Newton's method.

$$p_0 = 1.5$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1.5 - \frac{2\cdot\left(\frac{3}{2}\right)^2 - \frac{3}{2} - 2}{4\cdot\frac{3}{2} - 1}$$

$$= 1.3$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

$$= 1.3 - \frac{2\cdot(1.3)^2 - (1.3) - 2}{4\cdot(1.3) - 1}$$

$$= \cdots\cdots$$

4. Use the divided difference we have

x	$f[x]$	1st div. diff.	2nd.	3rd
$x_0 = -1$	1			
		1		
$x_0 = -1$	1		-2	
		-1		4
$x_1 = 0$	0		2	
		1		
$x_1 = 0$	0			

Then, the Hermite interpolation is

$$H(x) = 1 + 1\cdot(x - x_0) + (-2)(x - x_0)^2 + 4(x - x_0)^2(x - x_1)$$

$$= 1 + (x+1) - 2(x+1)^2 + 4(x+1)^2 x$$

5. (a) The Taylor expansion of $f(x_0-h)$ at x_0 is

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \frac{h^3}{3!}f'''(x_0) - \cdots$$

Then

$$M - N_1(h) = f'(x_0) - \frac{f(x_0) - f(x_0-h)}{h}$$

$$= f'(x_0) - \frac{\cancel{f(x_0)} - (\cancel{f(x_0)} - hf'(x_0) + \frac{h^2}{2!}f''(x_0) \cdots)}{h}$$

$$= \cancel{f'(x_0)} - (\cancel{f'(x_0)} - \frac{h}{2!}f''(x_0) + \cdots)$$

$$= \frac{h}{2!}f''(x_0) + \cdots$$

$$= O(h)$$

(b)
$$N_1(1) = \frac{f(x_0) - f(x_0-1)}{1} = \frac{f(1) - f(0)}{1} = \frac{0-1}{1} = -1$$

$$N_1(\tfrac{1}{2}) = \frac{f(x_0) - f(x_0-\frac{1}{2})}{\frac{1}{2}} = \frac{f(1) - f(\frac{1}{2})}{\frac{1}{2}} = \frac{0-0}{\frac{1}{2}} = 0$$

$$N_1(\tfrac{1}{4}) = \frac{f(x_0) - f(x_0-\frac{1}{4})}{\frac{1}{4}} = \frac{f(1) - f(\frac{3}{4})}{\frac{1}{4}} = \frac{0-(-\frac{1}{8})}{\frac{1}{4}} = \frac{1}{2}$$

By Richardson extrapolation,

$$N_2(h) = N_1(\tfrac{h}{2}) + [N_1(\tfrac{h}{2}) - N_1(h)] = 2N_1(\tfrac{h}{2}) - N_1(h)$$

$$N_3(h) = \tfrac{4}{3}N_2(\tfrac{h}{2}) - \tfrac{1}{3}N_2(h)$$

Therefore

$$N_2(\tfrac{1}{2}) = 2N_1(\tfrac{1}{4}) - N_1(\tfrac{1}{2}) = 1$$

$$N_2(1) = 2N_1(\tfrac{1}{2}) - N_1(1) = 1$$

and

$$N_3(1) = \tfrac{4}{3}N_2(\tfrac{1}{2}) - \tfrac{1}{3}N_2(1) = 1$$