Math 4513, Solution to Homework 2

1. Clearly, we have

$$
\begin{aligned}
\hat{p}_n - p &= p_n - p - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n} \\
&= (p_n - p) - \frac{((p_{n+1}-p)-(p_n-p))^2}{(p_{n+2}-p)-2(p_{n+1}-p)+(p_n-p)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\hat{p}_n - p}{p_n - p} &= \frac{p_n - p}{p_n - p} - \frac{((p_{n+1}-p)-(p_n-p))^2}{[(p_{n+2}-p)-2(p_{n+1}-p)+(p_n-p)](p_n-p)} \\
&= 1 - \frac{\dfrac{((p_{n+1}-p)-(p_n-p))^2}{(p_{n+1}-p)^2}}{\dfrac{[(p_{n+2}-p)-2(p_{n+1}-p)+(p_n-p)](p_n-p)}{(p_{n+1}-p)^2}} \\
&= 1 - \frac{\left(1-\frac{p_n-p}{p_{n+1}-p}\right)^2}{\left[\frac{p_{n+2}-p}{p_{n+1}-p}-2+\frac{p_n-p}{p_{n+1}-p}\right]\frac{p_n-p}{p_{n+1}-p}} \\
&= 1 - \frac{\left(1-\frac{1}{\lambda_n}\right)^2}{\left[\lambda_{n+1}-2+\frac{1}{\lambda_n}\right]\frac{1}{\lambda_n}}
\end{aligned}
$$

Since $\lim_{n\to\infty}\lambda_n = \lambda$, we have

$$
\begin{aligned}
\lim_{n\to\infty}\frac{\hat{p}_n - p}{p_n - p} &= \lim_{n\to\infty}\left(1 - \frac{\left(1-\frac{1}{\lambda_n}\right)^2}{\left[\lambda_{n+1}-2+\frac{1}{\lambda_n}\right]\frac{1}{\lambda_n}}\right) \\
&= 1 - \frac{\left(1-\frac{1}{\lambda}\right)^2}{\left[\lambda-2+\frac{1}{\lambda}\right]\frac{1}{\lambda}} \\
&= 1 - \frac{\left(1-\frac{1}{\lambda}\right)^2\lambda^2}{\left[\lambda-2+\frac{1}{\lambda}\right]\frac{1}{\lambda}\lambda^2} \\
&= 1 - \frac{(\lambda-1)^2}{\lambda^2-2\lambda+1} \\
&= 1 - 1 = 0.
\end{aligned}
$$

2. First, we use the Horner's method fo compute $P(2)$.

	Coefficient of x^5	Coefficient of x^4	Coefficient of x^3	Coefficient of x^2	Coefficient of x	Constant term
$x_0 = 2$	$a_5 = 1$	$a_4 = -1$	$a_3 = 2$	$a_2 = -3$	$a_1 = 1$	$a_0 = -4$
		$b_5x_0 = 2$	$b_4x_0 = 2$	$b_3x_0 = 8$	$b_2x_0 = 10$	$b_1x_0 = 22$
	$b_5 = 1$	$b_4 = 1$	$b_3 = 4$	$b_2 = 5$	$b_1 = 11$	$b_0 = 18$

Therefore,

$$P(x) = (x-2)(x^4 + x^3 + 4x^2 + 5x + 11) + 18 = (x-2)Q(x) + 18$$

where $Q(x) = x^4 + x^3 + 4x^2 + 5x + 11$. Clearly $P(2) = 18$. Furthermore, we have

$$P'(x) = (x-2)'Q(x) + (x-2)Q'(x) = Q(x) + (x-2)Q'(x)$$

and therefore $P'(2) = Q(2)$.

Now, apply the Horner's method again to compute $Q(2)$.

	Coefficient of x^4	Coefficient of x^3	Coefficient of x^2	Coefficient of x	Constant term
$x_0 = 2$	$a_4 = 1$	$a_3 = 1$	$a_2 = 4$	$a_1 = 5$	$a_0 = 11$
		$b_4x_0 = 2$	$b_3x_0 = 6$	$b_2x_0 = 20$	$b_1x_0 = 50$
	$b_4 = 1$	$b_3 = 3$	$b_2 = 10$	$b_1 = 25$	$b_0 = 61$

We have $P'(2) = Q(2) = 61$.

3. Use the Newton's divided difference, we have

x	$f(x)$	1st. div. diff.	2nd. div. diff.	3rd. div. diff.	4th. div. diff.
-1	15				
		-10			
0	5		4		
		-2		0	
1	3		4		1
		6		4	
2	9		16		
		38			
3	47				

By the Newton's divided difference formula, we have

$$\begin{aligned} P(x) &= 15 - 10(x+1) + 4(x+1)(x-0) + 0(x+1)(x-0)(x-1) + 1(x+1)(x-0)(x-1)(x-2) \\ &= x^4 - 2x^3 + 3x^2 - 4x + 5 \end{aligned}$$