Math 4513, SOlution to Homework 1

1. (a) Recall that the Taylor expansions for $\sin x$ and $\cos x$ at 0 are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

We have

$$\begin{aligned}
\frac{\sin h - h\cos h}{h} &= \frac{\left(h - \frac{h^3}{3!} + \frac{h^5}{5!} + \cdots\right) - h\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots\right)}{h} \\
&= \frac{\frac{h^3}{3} - h^5\left(\frac{1}{4!} - \frac{1}{5!}\right) + \cdots}{h} \\
&= \frac{h^2}{3} + O(h^4).
\end{aligned}$$

Hence the convergence rate is $O(h^2)$.

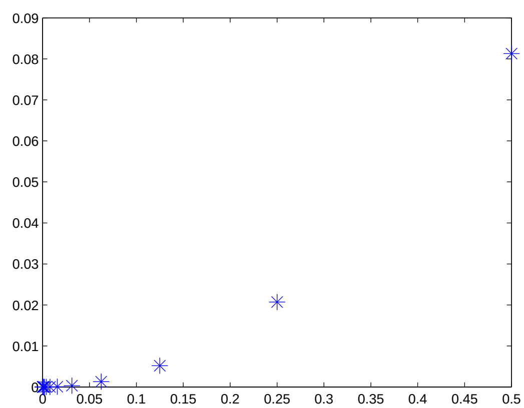
(b) The following matlab commands define the vectors

```
h = 2.^(-(1:20));
f = (sin(h)-h.*cos(h))./h;
```

Now plot the graph of f vesus h:

```
figure(1);
plot(h,f,'*', 'MarkerSize',12);
```

The is shown below:



(c) Since we know that the convergence rate is $O(h^2)$, from part (a). It is fine to use polynomials of degree greater than or equal to 2 in `polyfit`. The command

```
p = polyfit(h,f,2)
```

gives

```
p =

     0.3207    0.0023   -0.0000
```

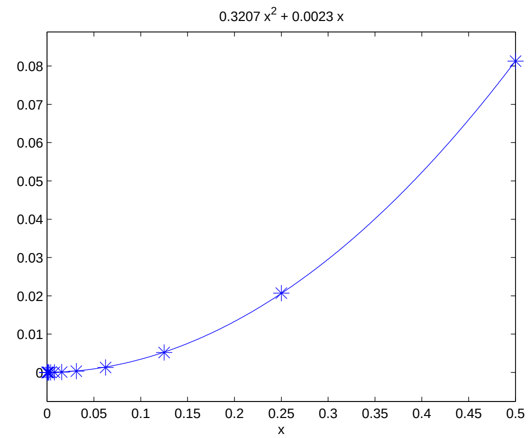
These are coefficients of the fitting polynomial. In other words, the polynomial is

$$p(x) = 0.3207x^2 + 0.0023x - 0.0000$$

(d) Now lets add the polynomial to the graph:

```
figure(2);
plot(h,f,'*', 'MarkerSize',12);
hold on;
ezplot('0.3207*x^2 + 0.0023*x', [0, 0.5]);
hold off
```

And the graph is shown below:



2. (a) The matlab problem and its output is given below:

```
TOL = 1e-8;
N=100;
for r=[2.6, 3.2, 3.5, 3.8, 4.2]
  a = 0.5;
  b = 1;
  i = 1;
  while ( i<=N & b-a>2*TOL )
    i = i+1;
    p = (a+b)/2;
    if  -r*p^2+(r-1)*p>0
      a = p;
    else
      b = p;
    end
  end
  fprintf('r = %4.2f, Exact sol = %12.10f\n', r, (r-1)/r);
  if  ( b-a<2*TOL )
```

```
    fprintf('  Found the root %12.10f after %d iterations.\n\n', p, i-1);
  else
    fprintf('  Method failed after %d iterations\n\n', i-1);
  end
end
```

and the outputs:

```
r = 2.60, Exact sol = 0.6153846154
  Found the root 0.6153846234 after 25 iterations.

r = 3.20, Exact sol = 0.6875000000
  Found the root 0.6874999851 after 25 iterations.

r = 3.50, Exact sol = 0.7142857143
  Found the root 0.7142857164 after 25 iterations.

r = 3.80, Exact sol = 0.7368421053
  Found the root 0.7368421108 after 25 iterations.

r = 4.20, Exact sol = 0.7619047619
  Found the root 0.7619047612 after 25 iterations.
```

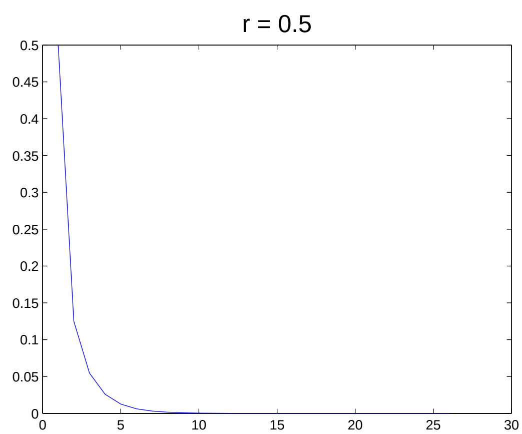
(b) The matlab program is

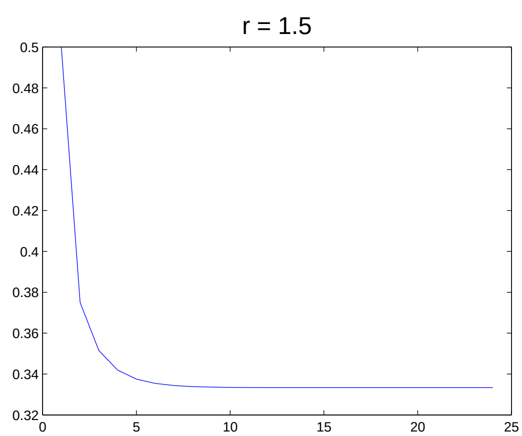
```
TOL = 1e-8;
N = 100;
figInd = 3;
for r = [0.5, 1.5, 2.6, 3.2, 3.5, 3.8, 4.2]
  x = zeros(100,1);
  x(1) = 0.5;
  i = 1;
  err = 1;
  while ( i<=N & err>TOL )
    i = i+1;
    x(i) = r*x(i-1)-r*x(i-1)^2;
    err = abs( x(i-1)-x(i) );
  end
  fprintf('r = %4.2f, Exact sol = %12.10f\n', r, max(0,(r-1)/r));
  if ( err<TOL )
    fprintf('  Found the root %12.10f after %d iterations.\n\n', x(i), i-1);
  else
    fprintf('  Method failed after %d iterations\n\n', i-1);
  end
  figure(figInd);
  plot(x(1:i));
  title(['r = ', num2str(r)],'FontSize',18);
  figInd = figInd + 1;
end
```

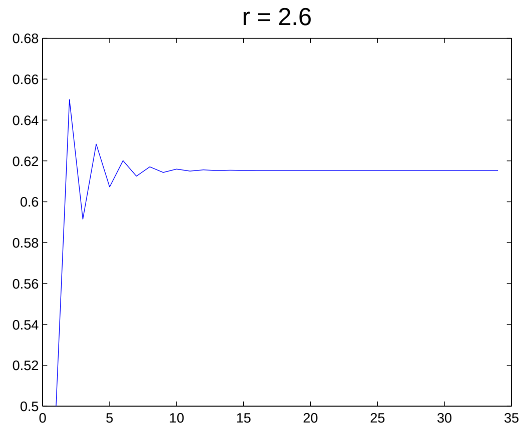
The out puts are

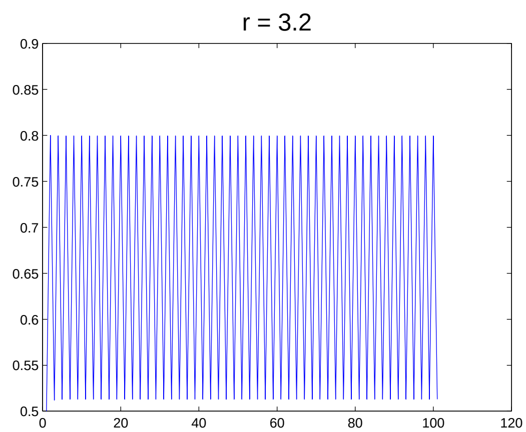
```
r = 0.50, Exact sol = 0.0000000000
  Found the root 0.0000000059 after 25 iterations.

r = 1.50, Exact sol = 0.3333333333
  Found the root 0.3333333411 after 23 iterations.

r = 2.60, Exact sol = 0.6153846154
  Found the root 0.6153846183 after 33 iterations.

r = 3.20, Exact sol = 0.6875000000
  Method failed after 100 iterations

r = 3.50, Exact sol = 0.7142857143
  Method failed after 100 iterations

r = 3.80, Exact sol = 0.7368421053
  Method failed after 100 iterations

r = 4.20, Exact sol = 0.7619047619
  Method failed after 13 iterations
```

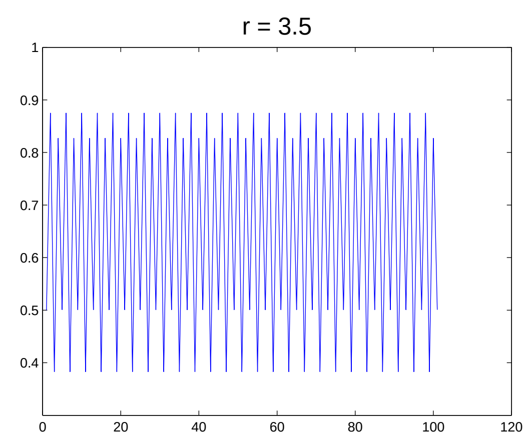
The graphs are given below:

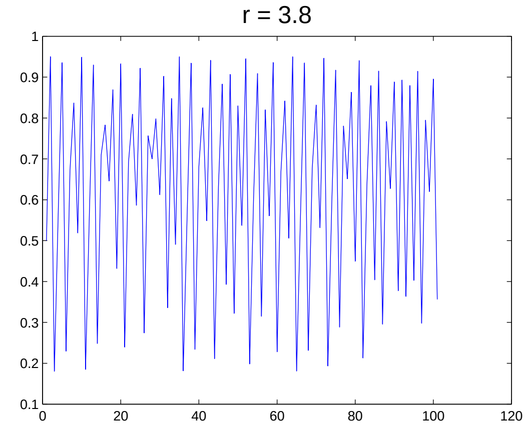


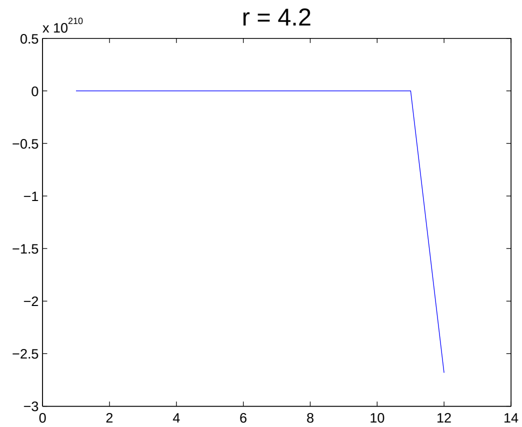



r = 2.6
0.68
0.66
0.64
0.62
0.6
0.58
0.56
0.54
0.52
0.5
0
5
10
15
20
25
30
35


r = 3.2
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
20
40
60
80
100
120


r = 3.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0
20
40
60
80
100
120


r = 3.8
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
20
40
60
80
100
120


r = 4.2
x 10^210
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
0
2
4
6
8
10
12
14