Math 4513, Introduction

1. Contents

We will cover the following chapters in the textbook:

Numerical Analysis By Burden and Faires

- Ch 1 Mathematical Preliminaries and Error Analysis
- Ch 2 Solutions of Equations in One variable
- Ch 3 Interpolation and Polynomial Approximation
- Ch 4 Numerical Differentiation and Integration
- Ch 5 Initial-Value Problems for Ordinary Differential Equations
- Ch 6 Direct Methods for Solving Linear Systems
- Ch 7 Iterative Techniques in Matrix Algebra (if time allows)

2. Review of Calculus (Section 1.1 in the textbook)

- Limits and Continuity
- Differentiability (and theorems)
- Integration
- Taylor Polynomials and Series

Limits and Continuity

Definition 1.1 A function f defined on a set X of real numbers has the limit L at x_0, written $\lim_{x \to x_0} f(x) = L$, if, given any real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$, whenever $x \in X$ and $0 < |x - x_0| < \delta$.

Definition 1.2 Let f be a function defined on a set x of real numbers and $x_0 \in X$. Then f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Furthermore, f is continuous on the set X if it is continuous at each number in X.

Limits and Continuity

Definition 1.3 Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. This sequence has the *limit* x (*converges to* x) if, for any $\varepsilon > 0$ there exists a positive integer N such that $|x_n - x| < \varepsilon$, whenever $n > N$. We denote it by

$$\lim_{n\to\infty} x_n = x \qquad \text{or} \qquad x_n \to x \text{ as } n \to \infty$$

Theorem 1.4 If f is a function defined on a set X of real numbers and $x_0 \in X$, then the following statements are equivalent:

(1) f is continuous at x_0.

(2) If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X converging to x_0, then $\lim_{x\to x_0} f(x) = f(x_0)$.

Differentiability

Definition 1.5 let f be a function in an open interval containing x_0. The function f is differentiable at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is called the derivative of f at x_0. A function that has a derivative at each number in a set X is differentiable on X.

Theorem 1.6 If a function f is differentiable at x_0, then f is continuous at x_0.

Differentiability

Theorem 1.7 (Rolle's Theorem) Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then there exists a number c in (a,b) such that $f'(c) = 0$.

Theorem 1.8 (Mean Value Theorem) If $f \in C[a,b]$ and f is differentiable on (a,b), then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 1.9 (Extreme Value Theorem) If $f \in C[a,b]$, then there exist $c_1,\ c_2 \in [a,b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ for all $x \in [a,b]$. In addition, if f is differentiable on (a,b), then th numbers c_1 and c_2 occur either at the endpoints of $[a,b]$ or where f' is zero.

Differentiability

Theorem 1.10 (Generalized Rolle's Theorem) Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If $f(x) = 0$ at $n + 1$ distinct numbers $a \leq x_0 < x_1 \cdots < x_n \leq b$, then there exists a number c in (x_0, x_n), and hence also in (a, b) such that $f^{(n)}(c) = 0$.

Theorem 1.11 (Intermediate Value Theorem) If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = K$.

Integration

Definition 1.12 The Riemann integral of the function f on the interval $[a, b]$ is the following limit, provided it exists:

$$\int_a^b f(x)dx = \lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} f(z_i)\Delta x_i$$

where the numbers $x_0, x_1, \ldots, x_n$ satisfy $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$, where $\Delta x_i = x_i - x_{i-1}$ for each $i = 1, 2, \ldots, n$, and z_i is arbitarily chosen in the interval $[x_{i-1}, x_i]$.

Theorem 1.13 (Weighted Mean Value Theorem for Integrals) Suppose $f \in C[a, b]$, the Riemann integral of g exists on $[a, b]$, and $g(x)$ does not change sign on $[a, b]$. Then there exists a number $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

Taylor Polynomials and Series

Theorem 1.14 (Taylor's Theorem) Suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number ξ between x_0 and x with

$$f(x) = P_n(x) + R_n(x)$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \frac{f^n(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \end{aligned}$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Here $P_n(x)$ is called the nth Taylor polynomial for f about x_0, and $R_n(x)$ is called the remainder term (or truncation error) associated with $P_n(x)$.

The infinite series obtained by taking the limit of $P_n(x)$ as $n \to \infty$ is called the Taylor series for f about x_0.

In the case $x_0 = 0$, the Taylor polynomial is often called a Maclaurin polynomial, and the Taylor series is often called a Maclaurin series.