Math 3013 Linear Algebra, Fall 2012
Quiz 2

Sept. 14, 2012

1. Show that $\mathbb{R}^3 = \textbf{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$

Solution For any vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$, consider the following problem:

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} \quad \Rightarrow \quad \begin{cases} c_1 + c_2 - c_3 = a \\ c_1 + c_2 + c_3 = b \\ c_1 - c_2 + c_3 = c \end{cases}$$

We only need to show that the above equation is always consistent, which means the vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ can always be expressed as a linear combination of $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$. By using Gaussian elimination

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & a\\ 1 & 1 & 1 & b\\ 1 & -1 & 1 & c\end{array}\right] \xrightarrow{r2-r1, r3-r1} \left[\begin{array}{ccc|c}1 & 1 & -1 & a\\ 0 & 0 & 2 & b-a\\ 0 & -2 & 2 & c-a\end{array}\right] \xrightarrow{switch\ r2\&r3} \left[\begin{array}{ccc|c}1 & 1 & -1 & a\\ 0 & -2 & 2 & c-a\\ 0 & 0 & 2 & b-a\end{array}\right]$$

Note the resulting system is always consistent since $rank(A) = rank(A|b) = 3$. This means $\begin{bmatrix}a\\b\\c\end{bmatrix}$ is a linear combination of $\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$. Thus $\mathbb{R}^3 = \textbf{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$.

2. Is the following set of vectors linearly dependent or linearly independent? If they are linearly dependent, find a dependence relationship among them.

$$\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$$

Solution Consider the following system

$$c_1\begin{bmatrix}0\\1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\1\\3\end{bmatrix} + c_3\begin{bmatrix}2\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Rightarrow \quad \begin{cases} 2c_2 + 2c_3 = 0 \\ c_1 + c_2 = 0 \\ 2c_1 + 3c_2 + c_3 = 0 \end{cases}$$

If the above system has only trivial solution, then these three vectors are linearly independent. Otherwise, they are linearly dependent. By the Gaussian elimination,

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] \xrightarrow{switch\ r1\&r2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] \xrightarrow{r3-2r1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r3-\frac{1}{2}r2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice c_3 is a free variable, which means there are nontrivial solutions. Hence the three vectors are linearly dependent. To find a dependence relationship, we need to write out the general solution for the above system. Let $c_3 = t$, then we have

$$\begin{cases} c_1 + c_2 = 0 \\ 2c_2 + 2c_3 = 0 \\ 0 = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} c_1 = t \\ c_2 = -t \\ c_3 = t \end{cases}$$

For example, when $t = 1$, we have

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$