Math 3013, Exam III, Nov. 28, 2012

Name: ____________________

Score:

The total is 50 points. Problem 1-3 are worth 4 points each.

1. (b) Which of the following statements is NOT true?

   (a) If 0 is an eigenvalue of matrix A, then A is not invertible.

   (b) The reduced row echelon form of a square matrix A has the same eigenvalues as A.

   (c) If A and B are $n \times n$ orthogonal matrices, then AB is also orthogonal.

   (d) If λ is an eigenvalue of A, then λ^2 must be an eigenvalue of A^2.

   (e) If A is an orthogonal matrix, then its rows form an orthonormal set of vectors.

2. (d) If $det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -3$, then $det \begin{bmatrix} a+2g & b+2h & c+2i \\ g & h & i \\ d & e & f \end{bmatrix} =?$

   $(a)\ -6 \qquad (b)\ 6 \qquad (c)\ -3 \qquad (d)\ 3 \qquad (e)\ 0$

3. (e) Which one of the following sets of vectors is NOT orthogonal?

   (a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \end{bmatrix};$

   (b) $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix};$

   (c) $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix};$

   (d) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix};$

   (e) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

4. (6 points) Does there exist k such that the following matrix is an orthogonal matrix? If yes, find all values of k that makes A orthogonal.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{k}{2} \\ -\frac{k}{2} & \frac{1}{2} \end{bmatrix}$$

Solution By definition, the columns of an orthogonal matrix are orthonormal. So we have

$$\begin{bmatrix} \frac{1}{2} \\ -\frac{k}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{k}{2} \\ \frac{1}{2} \end{bmatrix} = 0 \qquad \Rightarrow \qquad k \text{ can be any number}$$

$$\left\| \begin{bmatrix} \frac{1}{2} \\ -\frac{k}{2} \end{bmatrix} \right\| = 1 \qquad \Rightarrow \qquad k = \pm\sqrt{3}$$

$$\left\| \begin{bmatrix} \frac{k}{2} \\ \frac{1}{2} \end{bmatrix} \right\| = 1 \qquad \Rightarrow \qquad k = \pm\sqrt{3}$$

Therefore, when $k = \pm\sqrt{3}$, matrix A is orthogonal.

5. (8 points) It is known that matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ is diagonalizable. It has three eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$ and $\lambda_3 = 0$. We also know that the following vectors are eigenvectors of A, but not sure which one corresponds to which eigenvalue:

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution It is not hard to check that

$$A \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$. Similarly, one can show that $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = 0$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 3$.

Thus we can write

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Note the solution is not unique. There are other ways to write down D and P.

6. (8 points) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Is this matrix diagonalizable?

Solution By $det(A - \lambda I) = 0$, we have

$$det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 = 0$$

which implies $\lambda_1 = \lambda_2 = 3$ is a repeated eigenvalue of A.

Next, using $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we have

$$(A - 3I)\mathbf{x} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve this homogeneous equation, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This gives the eigenvector for A.

Matrix A is not diagonalizable, since the eigenvalue 3 has algebraic multiplicity 2 but geometric multiplicity 1.

7. (8 points) Compute the determinant of

$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$$

For which values of k is matrix A invertible?

Solution Note that

$$\begin{aligned} detA &= k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} - 0 + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix} \\ &= k(k^2+7) + k(-4k-3) \\ &= k(k^2-4k+4) \\ &= k(k-2)^2 \end{aligned}$$

Clearly, $detA = 0$ if and only if $k = 0,\ 2$. We know that a matrix A is invertible only when $detA \neq 0$, therefore A is invertible when $k \neq 0,\ 2$.

8. (8 points) Let W be a subspace spanned by the given vectors. Find a basis for $W^\perp$.

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution A vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in $W^\perp$ if

$$\mathbf{w}_1 \cdot \mathbf{x} = 0, \qquad \mathbf{w}_2 \cdot \mathbf{x} = 0,$$

that is,

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases}$$

Solve this homogeneous equation, we get

$$\mathbf{x} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,

$$\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for $W^\perp$.