Math 3013, Exam II, Oct. 22, 2012

Name: ____________________

Score:

The total is 50 points. Problem 1-3 are worth 4 points each.

1. (d) Given $A = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Which of the following statement is **not** true:

   (a) A is a reduced row echelon form

   (b) $rank(A) = 3$

   (c) $rank(A^T) = 3$

   (d) $nullity(A) = 3$

   (e) $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions

2. (c) Which of the following statement is **not** true?

   (a) Let $L: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation, then $L(\mathbf{0}) = \mathbf{0}$

   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^2$

   (c) For any given matrix A, we have $rank(A) = nullity(A^T)$

   (d) If A is a symmetric matrix, then $row(A) = col(A)$

   (e) Not all square matrices have LU factorization.

3. (d) The linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Then $L\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) =$

   $(a)\ \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\quad (b)\ \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $\quad (c)\ \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad (d)\ \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ $\quad (e)\ \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

4. (6 points) Is the following transformation a linear transformation? State the reason for your judgement.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$$

Solution T is not a linear transformation. We know that a linear transformation must satisfy

(a) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for any vectors $\mathbf{u}$ and $\mathbf{v}$,

(b) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any vector $\mathbf{v}$ and scalar c.

Indeed, the transformation T defined in this problem satisfies neither. Below we only show that T does not satisfy (b). Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Clearly

$$\begin{aligned} T(c\mathbf{v}) &= T\left(\begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}\right) = \begin{bmatrix} cv_1+1 \\ cv_2-1 \end{bmatrix} \\ cT(\mathbf{v}) &= c\begin{bmatrix} v_1+1 \\ v_2-1 \end{bmatrix} = \begin{bmatrix} c(v_1+1) \\ c(v_2-1) \end{bmatrix} = \begin{bmatrix} cv_1+c \\ cv_2-c \end{bmatrix} \end{aligned}$$

They are not equal to each other when $c \neq 1$. Therefore, T violates statement (b) and consequently it is not a linear transformation.

An alternative solution One can also use the following arguement to show that T is not a linear transformation. We know that all linear transformations are matrix transformations. If T is a linear transformation, there must exist a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$ for any vector $\mathbf{v}$. If $\mathbf{v}$ happens to be $\mathbf{0}$, then $T(\mathbf{0}) = A\mathbf{0} = \mathbf{0}$. However, according to the definition of T, we have

$$T(\mathbf{0}) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \mathbf{0}.$$

There is a contradiction. Thus, T can not be a linear transformation.

5. (8 points) Find the inverse of

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution Please check the solution to Quiz 3 for a detailed example for how to compute the inverse of a given matrix. Here we only give the final answer:

$$A^{-1} = \begin{bmatrix} -3 & -3 & 5 \\ 2 & 2 & -3 \\ -7 & -6 & 10 \end{bmatrix}$$

6. (8 points) Compute the LU factorization of

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix}$$

Then use the LU factorization to solve $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

Solution Please check the solution to Quiz 3 for a detailed example for how to compute the LU factorization. The steps are skipped here. Matrix A given in this problem has an LU factorization:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Next, we solve the linear system using the LU factorization.

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

Define $\mathbf{y} = U\mathbf{x}$. The above equation becomes

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

Using forward substitution, one can solve for $\mathbf{y}$,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$$

Now, by the definition of $\mathbf{y}$, we have

$$U\mathbf{x} = \mathbf{y} \quad \Rightarrow \quad \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$$

Using backward substitution, one can solve for $\mathbf{x}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

7. (8 points) Draw a digraph that has the given adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Find all 3-paths that go from vertex v_2 to vertex v_4?

Solution The graph is omitted. There are two 3-paths from v_2 to v_4,

(a) $v_2 \to v_1 \to v_2 \to v_4$

(b) $v_2 \to v_4 \to v_4 \to v_4$

8. (8 points) Find bases for $row(A)$, $col(A)$ and $null(A)$.

$$A = \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \end{bmatrix}$$

Solution By using Gaussian elimination, one can find a row echelon form for A

$$\begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The non-zero rows of the row echelon form form a basis for $row(A)$

$$\{[1\ -2\,1\,4\,4],\ [0\,0\,2\,6\,7]\}$$

(b) Find the columns of the row echelon form that contain a leading entry. Then the corresponding columns of A form a basis for $col(A)$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) To find a basis for $null(A)$, we first solve the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Using Gaussion elimination, clearly the augmented matrix has the row echelon form

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 4 & 4 & 0 \\ 0 & 0 & 2 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There are three free variables, x_2, x_4 and x_5. Set

$$x_2 = s, \qquad x_4 = t, \qquad x_5 = k$$

and then solve the homogeneous equation. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - t - \frac{1}{2}k \\ s \\ -3t - \frac{7}{2}k \\ t \\ k \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix}$$

This gives a basis for $null(A)$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$