Math 3013, Exam I, Sept. 19, 2012

Name: ___________________

Score:

The total is 50 points. Problem 1-3 are worth 4 points each.

1. (d) Which of the following statement is true?

   (a) The planes $x + 2y - z = 10$ and $4x + 8y - 2z = 3$ are parallel to each other.

   (b) $\mathbb{R}^2 = span\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}\}$.

   (c) The line $x = 1 - t,\ y = 2 + 3t,\ z = -2 - t$ goes through point $(-1,\ 3,\ -1)$

   (d) Vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -5 \\ -2 \end{bmatrix}$ are perpendicular to each other.

   (e) None of the above.

2. (e) For what value of k does the following linear system have a unique solution?

$$\begin{cases} x + 4y = 2 \\ x + k^2 y = k \end{cases}$$

$(a)\ k = 2$ $\quad (b)\ k = -2$ $\quad (c)\ k \neq 2$ $\quad (d)\ k \neq -2$ $\quad (e)\ k \neq 2$ and $k \neq -2$

3. (b) Given matrices:

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -3 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 9 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

which of the following expressions does **not** exist?

$(a)\ ABC$ $\quad (b)\ C^T D$ $\quad (c)\ CB$ $\quad (d)\ A^T + BC$ $\quad (e)\ D^T A$

4. (6 points) Give the parametric equation of a line passing through points $P = (1,\ 1,\ 1)$ and $Q = (4,\ 0,\ 2)$.

   Solution The direction vector is

   $$\mathbf{v} = (4, 0, 2) - (1, 1, 1) = (3, -1, 1)$$

   Therefore, we can write a parametric equationas follows:

   $$\begin{cases} x = 1 + 3t \\ y = 1 - t \\ z = 1 + t \end{cases}$$

5. (8 points) Solve the following system using Gaussian elimination.

$$\begin{cases} x + 2y + z = 3 \\ 3x + 7y + 2z = 11 \\ 2x + 6y = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 7 & 2 & 11 \\ 2 & 6 & 0 & 10 \end{array}\right] \xrightarrow{r2-3r1,\, r3-2r1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right] \xrightarrow{r3-2r2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent, and has one free variable z. Set $z = t$, we have

$$\begin{cases} x = -1 - 3t \\ y = 2 + t \\ x = t \end{cases}$$

6. (8 points) Determine whether the following set of vectors is linearly independent or linearly dependent. If they are linearly dependent, write down a dependency relationship.

$$\begin{bmatrix}1\\4\\3\\0\end{bmatrix}, \quad \begin{bmatrix}2\\3\\4\\5\end{bmatrix}, \quad \begin{bmatrix}3\\2\\2\\4\end{bmatrix}$$

Solution Consider the linear system

$$c_1\begin{bmatrix}1\\4\\3\\0\end{bmatrix} + c_2\begin{bmatrix}2\\3\\4\\5\end{bmatrix} + c_3\begin{bmatrix}3\\2\\2\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Next, we use Gaussian elimination to solve this system.

$$\left[\begin{array}{ccc|c}1&2&3&0\\4&3&2&0\\3&4&2&0\\0&5&4&0\end{array}\right] \xrightarrow{r2-4r1,\, r3-3r1} \left[\begin{array}{ccc|c}1&2&3&0\\0&-5&-10&0\\0&-2&-7&0\\0&5&4&0\end{array}\right] \xrightarrow{r2/(-5)} \left[\begin{array}{ccc|c}1&2&3&0\\0&1&2&0\\0&-2&-7&0\\0&5&4&0\end{array}\right]$$

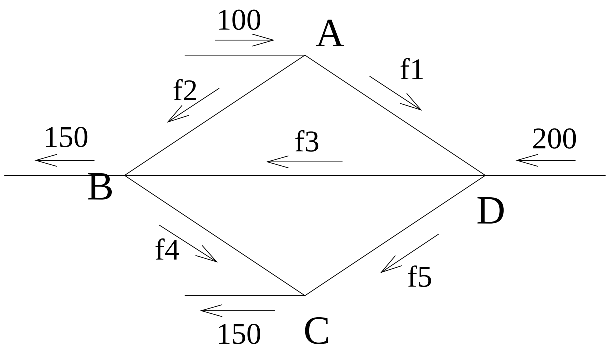
$$\xrightarrow{r3+2r1,\, r4-5r1} \left[\begin{array}{ccc|c}1&2&3&0\\0&1&2&0\\0&0&-3&0\\0&0&-6&0\end{array}\right] \xrightarrow{r4-2r3} \left[\begin{array}{ccc|c}1&2&3&0\\0&1&2&0\\0&0&-3&0\\0&0&0&0\end{array}\right]$$

This system is always consistent. Notice there is no free variable, which means it has a unique solution

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$$

Therefore, the three given vectors are linearly independent.

7. (8 points) Set up a linear system which describes the following water flow network using variables f_1, f_2, f_3, f_4 and f_5. You do NOT need to solve the system.



Solution We have

$$\begin{aligned} \text{at point A:} &\quad 100 = f_1 + f_2 \\ \text{at point B:} &\quad f_2 + f_3 = 150 + f4 \\ \text{at point C:} &\quad f_4 + f_5 = 150 \\ \text{at point D:} &\quad f_1 + 200 = f_3 + f_5 \end{aligned}$$

Combine the above,

$$\begin{cases} f_1 + f_2 = 100 \\ f_2 + f_3 - f_4 = 150 \\ f_4 + f_5 = 150 \\ f_1 - f_3 - f_5 = -200 \end{cases}$$

8. (8 points) Given

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & -2 & 5 \\ -2 & 8 & 6 \\ 5 & 6 & 6 \\ 3 & 4 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Compute AB^TC.

Solution For simplicity, we compute B^TC first and then $A(B^TC)$.

$$\begin{aligned} AB^TC &= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & -2 & 5 & 3 \\ -2 & 8 & 6 & 4 \\ 5 & 6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ -8 & 4 \\ -1 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -4 \\ -10 & 15 \\ -8 & 4 \end{bmatrix} \end{aligned}$$