Modelling with First Order Equations

1. Population Model

The population of mosquitoes in a certain area increases in a daily rate propotional to its current population, i.e., $0.01P$, where P is the current population. There are 200,000 mosquitoes in the area initially and predators (birds, bats, etc.) eat 20,000 mosquitoes a day. Determine the population of mosquitoes.

Step 1: Let P(t) be the population at time t. If there's no predator,

$$\frac{dP}{dt}=0.01\,P$$

Step 2: Since the predators eat 20,000 mosquitoes a day, we have

$$\frac{dP}{dt}=0.01\,P-20{,}000$$

Step 3: Initial condition

$$P(0)=200{,}000$$

2. Newton's Law of Cooling

A thermometer reads 36 degree when it is moved to a 70 degree room. 5 minutes later it reads 50 degree. Find the thermometer reading after 10 minutes.

Step 1: By Newton's Law of cooling, the temperature, T, of an object

$$\frac{dT}{dt} = k(E - T)$$

E is the enviromental temperature and k is the conducting rate.

Step 2: In our case, E=70. So

$$\frac{dT}{dt} = k(70 - T)$$

And the initial condition is

$$T(0) = 36$$

Step 3: Solver the equation: $$\frac{dT}{dt}=k(70-T)$$

$$T(0)=36$$

General solution: $$T=70+Ce^{-kt}$$

Plug in the initial condition: $$36=70+C \longrightarrow C=-34$$

Solution: $$T=70-34e^{-kt}$$

Step 4: Notice that T(5)=50,

$$50=70-34e^{-5k} \longrightarrow k=\frac{-1}{5}\ln\frac{10}{17}$$

Therefore, $$T(10)=70-34e^{(1/5\ln(10/17))*10}\approx 58.2353$$

3. Escape Velocity

A rocket is launched from the surface of the earth at a launching speed v(0). Find the least initial speed for which the rocket will not return to the earth.

Step 1: The only force on the rocket is the gravity. By Newton's Universal Law of Gravitation, the gravitational force is

$$\frac{-mgR^2}{(x+R)^2}$$

R =earth's radius, and x is the height of the rocket above ground.

Step 2: Newton's Second Law

$$m\frac{dv}{dt}=\frac{-mgR^2}{(x+R)^2}$$

Step 3: Unknowns are t, x, v (Too many variables!)

Let's eliminate t by: $$\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$$

Then: $$m\,v\frac{dv}{dx}=\frac{-mgR^2}{(x+R)^2} \longrightarrow v\frac{dv}{dx}=\frac{-gR^2}{(x+R)^2}$$

It is seperable and the general solution is:

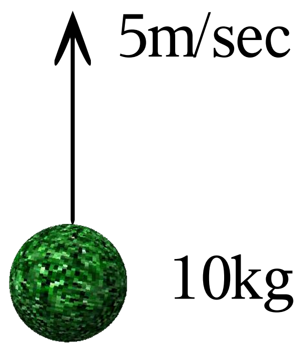
$$\frac{v^2}{2}=\frac{gR^2}{x+R}+C$$

Step 4: To escape means when v=0, x =infinity. Therefore, C=0.

The escape volocity is:

$$\frac{v(0)^2}{2}=\frac{gR^2}{0+R} \longrightarrow v(0)=\sqrt{2gR}$$

4. Throwing a ball



A 10kg ball is thrown upward at 5m/sec initial speed from a table which is 2m high. The force due to air resistance is 2v.

(1) find the maximum height above the ground that the ball reaches.

(2) Assuming the ball missed the table on the way down, find the time that it hit the ground.

Step 1: Recall the equation for a falling object:

$$m\frac{dv}{dt}=mg-2v$$

We have:

$$\frac{dv}{dt}=9.8-\frac{1}{5}v$$

$$v(0)=-5$$

Step 2: Solve the equation gives

$$v = 49 - 54e^{-t/5}$$

Step 3: When the ball reaches the highest point, v=0. So

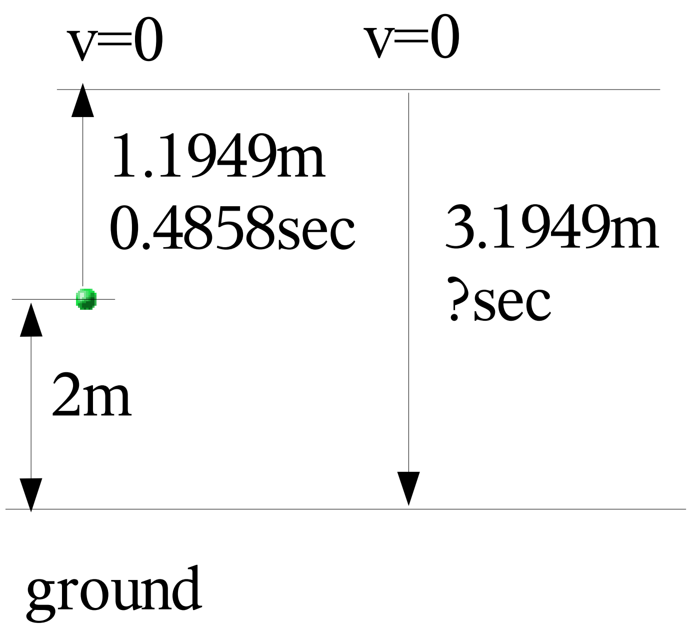
$$0 = 49 - 54e^{-t/5} \quad \longrightarrow \quad t \approx 0.4858$$

The height of the ball above the table is

$$\left|\int_0^{0.4858} v\,dt\right| = \left|\int_0^{0.4858} (49 - 54e^{-t/5})\,dt\right| \approx 1.1949$$

The height of the ball above the ground is

$$2 + 1.1949 = 3.1949$$



Step 4: Since the height of the ball satisfies

$$\frac{dx}{dt} = -v$$

Therefore

$$x = \int -v\,dt = \int (-49 + 54e^{-t/5})\,dt$$

$$x = -49t - 270e^{-t/5} + C$$

Notice that when t=0, x=2. So

$$2 = -270 + C \longrightarrow C = 272$$

Therefore, when the ball hit the ground, (x=0), we have

$$0 = -49t - 270e^{-t/5} + 272 \longrightarrow t \approx 1.3156$$