Quiz # 2– Math 2233, Differential Equations – Sep. 21, 2009

1. (4 points) Solve the initial value problem

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1,\ y'(0) = -1 \end{cases}$$

Solution From the characteristic equation

$$r^2 - r - 2 = 0 \qquad \Rightarrow \qquad (r+1)(r-2) = 0,$$

we have $r_1 = -1$ and $r_2 = 2$. Hence the general solution is $y = c_1 e^{-t} + c_2 e^{2t}$. Clearly $y' = -c_1 e^{-t} + 2c_2 e^{2t}$. Then by the initial condition, we have

$$\begin{cases} y(0) = c_1 + c_2 = 1 \\ y'(0) = -c_1 + 2c_2 = -1 \end{cases} \qquad \Rightarrow \qquad \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

Therefore, the particular solution is $y = e^{-t}$.

2. (6 points) Show that $cos\sqrt{t}$ and $sin\sqrt{t}$, where $t > 0$, are two solutions to the equation $ty'' + \frac{1}{2}y' + \frac{1}{4}y = 0$, $t > 0$. Furthermore, show that they form a fundamental set of solutions for this problem.

Solution Notice that

$$\begin{aligned} y_1 &= \cos\sqrt{t} & y_2 &= \sin\sqrt{t} \\ y_1' &= -\frac{1}{2}t^{-1/2}\sin\sqrt{t} & y_2' &= \frac{1}{2}t^{-1/2}\cos\sqrt{t} \\ y_1'' &= \frac{1}{4}t^{-3/2}\sin\sqrt{t} - \frac{1}{4}t^{-1}\cos\sqrt{t} & y_2'' &= -\frac{1}{4}t^{-3/2}\cos\sqrt{t} - \frac{1}{4}t^{-1}\sin\sqrt{t} \end{aligned}$$

Therefore

$$\begin{aligned} &ty_1'' + \frac{1}{2}y_1' + \frac{1}{4}y_1 \\ =&t\left(\frac{1}{4}t^{-3/2}\sin\sqrt{t} - \frac{1}{4}t^{-1}\cos\sqrt{t}\right) + \frac{1}{2}\left(-\frac{1}{2}t^{-1/2}\sin\sqrt{t}\right) + \frac{1}{4}\cos\sqrt{t} = 0 \end{aligned}$$

$$\begin{aligned} &ty_2'' + \frac{1}{2}y_2' + \frac{1}{4}y_2 \\ =&t\left(-\frac{1}{4}t^{-3/2}\cos\sqrt{t} - \frac{1}{4}t^{-1}\sin\sqrt{t}\right) + \frac{1}{2}\left(\frac{1}{2}t^{-1/2}\cos\sqrt{t}\right) + \frac{1}{4}\sin\sqrt{t} = 0 \end{aligned}$$

So both $cos\sqrt{t}$ and $sin\sqrt{t}$ are solutions. Finally, the Wronskian

$$\begin{aligned} W = y_1y_2' - y_1'y_2 &= \cos\sqrt{t}\left(\frac{1}{2}t^{-1/2}\cos\sqrt{t}\right) - \left(-\frac{1}{2}t^{-1/2}\sin\sqrt{t}\right)\sin\sqrt{t} \\ &= \frac{1}{2}t^{-1/2}\cos^2\sqrt{t} + \frac{1}{2}t^{-1/2}\sin^2\sqrt{t} \\ &= \frac{1}{2}t^{-1/2} \neq 0. \end{aligned}$$

Therefore $cos\sqrt{t}$ and $sin\sqrt{t}$ form a fundamental set of solutions.