Quiz # 6– Math 2233, Differential Equations – Oct. 8, 2008

1. Find the general solution to

$$y'' + 4y' + 4 = t^{-2}e^{-2t}, \quad t > 0$$

   Formula for variation of parameters:

$$Y = y_1(t)\int_{t_0}^{t} \frac{-y_2(s)g(s)}{W(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)}ds$$

Solution:

Step 1: Compute solutions y_1 and y_2 to $y''+4y'+4 = 0$. The charatcteristic equation is

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad r_1 = r_2 = -2$$

Therefore $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$.

Step 2: Compute a specific solution Y by the formula. First, the Wronskian is

$$W = y_1y_2' - y_2y_1' = e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t}(-2e^{-2t}) = e^{-4t}$$

The right-hand side $g(t) = t^{-2}e^{-2t}$. Therefore, by the formula,

$$\begin{aligned}
Y &= e^{-2t}\int_{t_0}^{t} \frac{-(se^{-2s})(s^{-2}e^{-2s})}{e^{-4s}}ds + te^{-2t}\int_{t_0}^{t} \frac{(e^{-2s})(s^{-2}e^{-2s})}{e^{-4s}}ds \\
&= e^{-2t}\int_{t_0}^{t} -s^{-1}ds + te^{-2t}\int_{t_0}^{t} s^{-2}ds \\
&= -e^{-2t}(\ln|s|)|_{t_0}^{t} + te^{-2t}(-s^{-1})|_{t_0}^{t} \\
&= -e^{-2t}(\ln|t| - \ln|t_0|) + te^{-2t}(-t^{-1} + t_0^{-1})
\end{aligned}$$

By setting $t_0 = 1$ and since $t > 0$, we have

$$Y = -e^{-2t}\ln t - e^{-2t} + te^{-2t}$$

(Note: t_0 can not be 0, otherwise $\ln|t_0|$ and t_0^{-1} are meaningless.)

Step 3: Combine the above, the general solution is

$$\begin{aligned}
y &= c_1y_1 + c_2y_2 + Y \\
&= c_1e^{-2t} + c_2te^{-2t} - e^{-2t}\ln t - e^{-2t} + te^{-2t} \\
&= C_1e^{-2t} + C_2te^{-2t} - e^{-2t}\ln t
\end{aligned}$$

where $C_1 = c_1 - 1$ and $C_2 = c_2 + 1$.