**Quiz # 10**– Math 2233, Differential Equations – Nov. 13, 2008

1. Solve the problem
$$\begin{cases} y'' + 4y = \delta(t-\pi) - \delta(t-2\pi) \\ y(0) = 0,\ y'(0) = 0 \end{cases}$$

Formula: $\mathcal{L}\{\delta(t-c)\} = e^{-cs}$

$$\mathcal{L}\{\sin at\} = \tfrac{a}{s^2+a^2}$$

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

**Solution.**

Step 1 Take Laplace transform on both sides of the equation:

$$\begin{aligned} & \mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\delta(t-\pi) - \delta(t-2\pi)\} \\ \Rightarrow\quad & s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = e^{-\pi s} - e^{-2\pi s} \\ \Rightarrow\quad & (s^2+4)\mathcal{L}\{y\} = e^{-\pi s} - e^{-2\pi s} \\ \Rightarrow\quad & \mathcal{L}\{y\} = \frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4} \end{aligned}$$

Step 2 Therefore

$$\begin{aligned} y &= \mathcal{L}^{-1}\{\frac{e^{-\pi s} - e^{-2\pi s}}{s^2+4}\} \\ &= \mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s^2+4}\} - \mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s^2+4}\} \end{aligned}$$

To compute $\mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s^2+4}\}$, we notice that

$$c = \pi, \quad F(s) = \frac{1}{s^2+4} \ \Rightarrow\ f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2+4}\} = \frac{1}{2}\sin 2t$$

Hence

$$\mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s^2+4}\} = u_\pi(t)f(t-\pi) = u_\pi(t)\frac{1}{2}\sin 2(t-\pi)$$

Similarly

$$\mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s^2+4}\} = u_{2\pi}(t)\frac{1}{2}\sin 2(t-2\pi)$$

Combine all the above, we have

$$y = u_\pi(t)\frac{1}{2}\sin 2(t-\pi) - u_{2\pi}(t)\frac{1}{2}\sin 2(t-2\pi)$$