Math 2233, Exam III, Nov. 20, 2008

Name: ____________________

Score:

Part 1: Multiple choice. Each question is worth 4 points.

1. (d) $L[y] = y'' - 3y' + 2y$. Evaluate $L[e^{-t}]$.

   (a) $-2e^{-t}$;

   (b) 0;

   (c) $4e^{-t}$;

   (d) $6e^{-t}$;

   (e) $8e^{-t}$.

2. (b) Find the Laplace transform of

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}.$$

   (a) $e^{s}(\frac{1}{s^2} + \frac{1}{s})$;

   (b) $e^{-s}(\frac{1}{s^2} + \frac{1}{s})$;

   (c) $e^{-s}(\frac{1}{s^2} - \frac{1}{s})$;

   (d) $\frac{e^{-s}}{s^2}$;

   (e) $\frac{e^{s}}{s^2}$.

3. (c) Find a proper form of the specific solution Y of the differential equation

$$y''' - 3y'' + 3y' - y = 3e^t.$$

(a) Ae^t;
(b) At^2e^t;
(c) At^3e^t;
(d) $(A\cos t + B\sin t)t^2e^t$;
(e) $(At + B)t^3e^t$.

4. (a) Which of the following is equal to the inverse Laplace transform of

$$\frac{1}{(s+1)(s^2+1)}$$

(a) $\int_0^t e^{-(t-\tau)} \sin\tau \, d\tau$;
(b) $\int_0^t e^{-(t+\tau)} \sin\tau \, d\tau$;
(c) $\int_0^t e^{t-\tau} \sin\tau \, d\tau$;
(d) $\int_0^t e^{t+\tau} \sin\tau \, d\tau$;
(e) $\int_0^t e^{t-\tau} \cos\tau \, d\tau$.

5. (e) Find the inverse Laplace transform of

$$\frac{s}{s^2 - 4s + 8}.$$

(a) $\frac{1}{2}e^{2t}\sin(2t)$;
(b) $e^{2t}\cos(2t)$;
(c) $\frac{1}{2}e^{2t}\cos(2t)$;
(d) $e^{2t}(\cos(2t) + 2\sin(2t))$;
(e) $e^{2t}(\cos(2t) + \sin(2t))$.

Part 2. Partial credit section. Show all your work neatly and concisely, and indicate your final answer clearly. If you simply write down the final answer without appropriate intermediate steps, you may not get full credit for that problem.

6. (8 points) Solve the following initial value problem:

$$\begin{cases} y''' - y'' + y' - y = 0 \\ y(0) = 2,\ y'(0) = 1,\ y''(0) = 0. \end{cases}$$

You final answer should NOT contain any integral.

Solution 1 The characteristic equation is

$$\begin{aligned} & r^3 - r^2 + r - 1 = 0 && \Rightarrow && r^2(r-1) + (r-1) = 0 \\ \Rightarrow\quad & (r^2+1)(r-1) = 0 && \Rightarrow && r_1 = 1, \quad r_2 = i \quad r_3 = -i. \end{aligned}$$

Hence the general solution is

$$y = C_1 e^t + C_2 \cos t + C_3 \sin t.$$

By the initial condition, we have

$$\begin{cases} C_1 + C_2 = 2 \\ C_1 + C_3 = 1 \\ C_1 - C_2 = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} C_1 = 1 \\ C_2 = 1 \\ C_3 = 0 \end{cases}$$

Therefore the particular solution is

$$y = e^t + \cos t.$$

Solution 2 Apply Laplace transform to the equation, we have

$$\begin{aligned} & \mathcal{L}\{y'''\} - \mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{y\} = 0 \\ \Rightarrow\quad & [s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0)] - [s^2\mathcal{L}\{y\} - sy(0) - y'(0)] \\ & \quad + [s\mathcal{L}\{y\} - y(0)] - \mathcal{L}\{y\} = 0 \\ \Rightarrow\quad & (s^3 - s^2 + s - 1)\mathcal{L}\{y\} = 2s^2 - s + 1 \\ \Rightarrow\quad & \mathcal{L}\{y\} = \frac{2s^2 - s + 1}{s^3 - s^2 + s - 1} = \frac{2s^2 - s + 1}{(s^2+1)(s-1)} \end{aligned}$$

Use partial fraction to get

$$\mathcal{L}\{y\} = \frac{a}{s-1} + \frac{bs + c}{s^2+1} \qquad \Rightarrow \qquad \mathcal{L}\{y\} = \frac{1}{s-1} + \frac{s}{s^2+1}$$

Hence by computing the inverse Laplace transform, we have

$$y = e^t + \cos t.$$

7. (8 points) Use Laplace transform to solve the following initial value problem:

$$y'' + 4y = 4 + \delta(t-3), \qquad \text{with } y(0) = 0 \text{ and } y'(0) = 0.$$

You final answer should NOT contain any integral.

Solution Apply Laplace transform to the equation, we have

$$\begin{aligned}
& [s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + 4\mathcal{L}\{y\} = \frac{4}{s} + e^{-3s} \\
\Rightarrow \qquad & (s^2+4)\mathcal{L}\{y\} = \frac{4}{s} + e^{-3s} \\
\Rightarrow \qquad & \mathcal{L}\{y\} = \frac{4}{s(s^2+4)} + \frac{e^{-3s}}{s^2+4} \\
& \qquad \text{(by partial fraction)} \\
\Rightarrow \qquad & \mathcal{L}\{y\} = \frac{a}{s} + \frac{bs+c}{s^2+4} + \frac{e^{-3s}}{s^2+4} \\
\Rightarrow \qquad & \mathcal{L}\{y\} = \frac{1}{s} - \frac{s}{s^2+4} + \frac{e^{-3s}}{s^2+4}
\end{aligned}$$

Now compute the inverse Laplace transform:

$$\begin{aligned}
y &= \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \mathcal{L}^{-1}\{\frac{e^{-3s}}{s^2+4}\} \\
&= 1 - \cos 2t + \frac{1}{2}\mathcal{L}^{-1}\{e^{-3s}\frac{2}{s^2+4}\} \\
&= 1 - \cos 2t + \frac{1}{2}u_3(t)\sin[2(t-3)].
\end{aligned}$$

8. (8 points) Find the solution $y(t)$ to the integral equation

$$y(t) + 2\int_0^t \cos(t-\tau)y(\tau)\,d\tau = \sin t.$$

Solution Apply Laplace transform to the equation, we have

$$\begin{aligned}
& \mathcal{L}\{y\} + 2\mathcal{L}\{\cos t * y\} = \frac{1}{s^2+1} \\
\Rightarrow \qquad & \mathcal{L}\{y\} + 2\mathcal{L}\{\cos t\}\cdot\mathcal{L}\{y\} = \frac{1}{s^2+1} \\
\Rightarrow \qquad & \mathcal{L}\{y\} + 2\frac{s}{s^2+1}\mathcal{L}\{y\} = \frac{1}{s^2+1} \\
& \text{(Multiply the equation by } s^2+1\text{)} \\
\Rightarrow \qquad & (s^2+1)\mathcal{L}\{y\} + 2s\mathcal{L}\{y\} = 1 \\
\Rightarrow \qquad & (s^2+2s+1)\mathcal{L}\{y\} = 1 \\
\Rightarrow \qquad & \mathcal{L}\{y\} = \frac{1}{s^2+2s+1} = \frac{1}{(s+1)^2}
\end{aligned}$$

Therefore

$$y = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} = te^{-t} \qquad \text{(this is a formula)}$$

9. (6 points) If $y' = xy^2 - y^3$ and $y(1) = -1$, find $y'(1)$ and $y''(1)$.

Solution Substitute $x = 1$ into the original equation, we have

$$y'(1) = 1 \cdot (-1)^2 - (-1)^3 = 2$$

Take derivative of the equation with respect to x, we have

$$y'' = (xy^2 - y^3)' = (xy^2)' - (y^3)' = (y^2 + x(2y)y') - 3y^2y'$$

Substitute $x = 1$ into the above equation, we have

$$\begin{aligned} y''(1) &= y(1)^2 + (1) \cdot (2y(1)) \cdot y'(1) - 3(y(1))^2 \cdot y'(1) \\ &= (-1)^2 + (1) \cdot (-2) \cdot (2) - 3(-1)^2 \cdot 2 \\ &= 1 - 4 - 6 \\ &= -9 \end{aligned}$$

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	t^n	$\dfrac{n\,!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\dfrac{n\,!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}\,F\left(\dfrac{s}{c}\right), \quad c > 0$
16.	$\displaystyle\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$