Math 2233, Exam II, Oct. 16, 2008

Name: ____________________

Score:

Formula for variation of parameters:

$$Y = y_1(t)\int_{t_0}^{t} \frac{-y_2(s)g(s)}{W(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)}ds$$

Part 1: Multiple choice. Each question is worth 4 points.

1. (c) Find the largest open interval for which the initial value problem

$$(t-2)y'' + \frac{y}{t-4} - \frac{t}{t-1} = 0, \quad y(3) = 0,\ y'(3) = 1,$$

has a solution.

(a) $0 < t < 4$;

(b) $1 < t < 4$;

(c) $2 < t < 4$;

(d) $1 < t < \infty$;

(e) $2 < t < \infty$.

2. (b) Which one is NOT a solution for the differential equation

$$y'' + 3y' + 2y = 0.$$

(a) e^{-t};

(b) e^{t};

(c) $-e^{-t}$;

(d) $-e^{-2t}$;

(e) $10e^{-t}$.

3. (c) Determine a proper form of the particular solution for the following equation if the method of undetermined coefficients is to be used.

$$y'' - 2y' = t + 4te^{2t} + e^{2t}\sin 2t.$$

(a) $At + B + Cte^{2t} + De^{2t} + Ee^{2t}\sin 2t + Fe^{2t}\cos 2t$;

(b) $At^2 + Bt + Cte^{2t} + De^{2t} + Ee^{2t}\sin 2t + Fe^{2t}\cos 2t$;

(c) $At^2 + Bt + Ct^2e^{2t} + Dte^{2t} + Ee^{2t}\sin 2t + Fe^{2t}\cos 2t$;

(d) $At^2 + Bt + Ct^2e^{2t} + Dte^{2t} + Ete^{2t}\sin 2t + Fte^{2t}\cos 2t$;

(e) $At^2 + Bt^2e^{2t} + Cte^{2t}\sin 2t$.

4. (a) The value of the constant r such that $y = x^r$ solves $x^2y'' + xy' - 2y = 0$ for $x > 0$ are:

(a) $\pm\sqrt{2}$;

(b) $\pm i\sqrt{2}$;

(c) $1 \pm \sqrt{2}$;

(d) $-1,\ -2$;

(e) $1,\ -2$.

5. (e) If one solution of $y'' + y' - 2y = f(x)$ is $y(x) = \ln x$, then the general solution is:

(a) $c_1 \ln x$;

(b) $c_1 \ln x + c_2e^{x} + c_3e^{-2x}$;

(c) $c_1e^{-x} + c_2e^{2x} + \ln x$;

(d) $c_1e^{x} + c_2e^{-2x}$;

(e) $c_1e^{x} + c_2e^{-2x} + \ln x$.

Part 2. Partial credit section. Show all your work neatly and concisely, and indicate your final answer clearly. If you simply write down the final answer without appropriate intermediate steps, you may not get full credit for that problem.

6. (8 points) Use the method of undetermined coefficients to find the general solution to

$$y'' - 2y' - 3y = e^t + 1.$$

Solution:

Step 1. Solve the associated homogeneous equation $y'' - 2y' - 3y = 0$. The characteristic equation is

$$r^2 - 2r - 3 = 0 \quad \Rightarrow (r+1)(r-3) = 0$$

Therefore $y_1 = e^{-t}$ and $y_2 = e^{3t}$

Step 2. By the method of undetermined coefficients

$$Y = Ae^t + B$$

Hence

$$Y' = Ae^t, \qquad Y'' = Ae^t$$

Substitute them into the differential equation, we have

$$\begin{aligned} & Ae^t - 2Ae^t - 3(Ae^t + B) = e^t + 1 \\ \Rightarrow \quad & -4Ae^t - 3B = e^t + 1 \\ \Rightarrow \quad & A = -1/4, \quad B = -1/3 \\ \Rightarrow \quad & Y = -\frac{1}{4}e^t - \frac{1}{3} \end{aligned}$$

Step 3. Combine the above, the general solution is

$$y = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{4}e^t - \frac{1}{3}$$

7. (a) (4 points) Show that $y_1 = t$ and $y_2 = t^{-1}$ are solutions to $t^2y'' + ty' - y = 0$.

(b) (2 points) Evaluate the Wronskian $W[t, t^{-1}]$.

(c) (2 points) Find the solution of the initial value problem $t^2y'' + ty' - y = 0$, $y(1) = 2$, $y'(1) = 4$.

Solution

(a) Notice that $y_1 = t$, $y_1' = 1$, $y_1'' = 0$, substitute them into the equation:

$$t^2 \cdot 0 + t \cdot 1 - t = 0 \quad \text{The equation holds}$$

Similarly $y_2 = t^{-1}$, $y_2' = -t^{-2}$, $y_2'' = 2t^{-3}$, substitute them into the equation:

$$t^2 \cdot 2t^{-3} + t \cdot (-t^{-2}) - t^{-1} = 2t^{-1} - t^{-1} - t^{-1} = 0 \quad \text{The equation holds}$$

Therefore bothe y_1 and y_2 are solutions.

(b)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = t \cdot (-t^{-2}) - 1 \cdot t^{-1} = -2t^{-1}$$

(c) By part (b), the Wronskian is not identically 0, hence the general solution is

$$y = c_1 t + c_2 t^{-1}$$

Clearly we have $y' = c_1 - c_2 t^{-2}$. By the initial conditions, we have

$$\begin{cases} 2 = y(1) = c_1 + c_2 \\ 4 = y'(1) = c_1 - c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 3 \\ c_2 = -1 \end{cases}$$

Hence the particular solution is

$$y = 3t - t^{-1} = 3t - \frac{1}{t}$$

8. (6 points) Find the general solution to

$$y'' - 2y' + y = \frac{e^t}{t^2+1}.$$

(Hint: $\int \frac{1}{t^2+1}\,dt = \arctan t + C$*)*

Solution

Step 1. Solve the homogeneous equation $y'' - 2y' + y = 0$.

$$r^2 - 2r + 1 = 0 \quad \Rightarrow \quad r_1 = r_2 = 1$$

Hence we have

$$y_1 = e^t, \qquad y_2 = te^t$$

Step 2. Compute a specific solution by the method of variation of parameters. First, the Wronskian is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^t(te^t + e^t) - e^t te^t = e^{2t}$$

Then by the fomula

$$\begin{aligned} Y &= e^t \int_{t_0}^{t} \frac{-se^s \frac{e^s}{s^2+1}}{e^{2s}} ds + te^t \int_{t_0}^{t} \frac{e^s \frac{e^s}{s^2+1}}{e^{2s}} ds \\ &= e^t \int_{t_0}^{t} \left(-\frac{s}{s^2+1}\right) ds + te^t \int_{t_0}^{t} \frac{1}{s^2+1} ds \\ &= e^t \left(-\frac{1}{2}\ln(s^2+1)\right)\Big|_{t_0}^{t} + te^t \arctan s\Big|_{t_0}^{t} \end{aligned}$$

(Note: $\int_{t_0}^{t}\left(-\frac{s}{s^2+1}\right) ds$ can be computed by change of variable $u = s^2 + 1$.)
By setting $t_0 = 0$ and notice that $\ln 1 = 0$, $\arctan 0 = 0$, we have

$$Y = -\frac{1}{2}e^t \ln(t^2+1) + te^t \arctan t$$

Step 3. Combine the above, the general solution is

$$y = c_1 e^t + c_2 te^t - \frac{1}{2}e^t \ln(t^2+1) + te^t \arctan t$$

9. (8 points) A mass of 100g stretches a spring 5cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no damping and no external forces, determine the position $u(t)$ of the mass at any time t. And find the natural frequency of vibration.
(Hint: The acceleration due to gravity is $9.8\, m/sec^2 = 980\, cm/sec^2$. We also know that $\sqrt{196} = 14$.)

Solution We will use the units g, cm and sec when solving this problem. The mechanic oscillation is given by

$$mu'' + \gamma u' + ku = F(t)$$

Because there's no damping and no external forces, we have $\gamma = 0$ and $F(t) = 0$. Now we need to find the values for m and k. It is clear that $m = 100$. To compute k, we use

$$mg = kL \quad \Rightarrow \quad 100 \times 980 = k \times 5 \quad \Rightarrow \quad k = \frac{98000}{5} = 19600$$

Hence the equation becomes

$$100u'' + 19600u = 0$$

By dividing the equation by 100, it can be simplified into

$$u'' + 196u = 0$$

Combine it with the initial conditions, we have

$$\begin{cases} u'' + 196u = 0 \\ u(0) = 0, \quad u'(0) = 10 \end{cases}$$

Now we need to solve the initial value problem. The characteristic equation is

$$r^2 + 196 = 0 \quad \Rightarrow \quad r = \pm\sqrt{-196} = \pm 14i$$

Therefore the general solution is

$$u = c_1 \cos 14t + c_2 \sin 14t$$

By using the initial condition,

$$\begin{cases} 0 = c_1 \cos 0 + c_2 \sin 0 \\ 10 = -14c_1 \sin 0 + 14c_2 \cos 0 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 0 \\ c_2 = \frac{10}{14} \end{cases}$$

Hence the particular solution is

$$u = \frac{10}{14} \sin 14t$$

Finally, the natural frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{19600}{100}} = \sqrt{196} = 14.$$