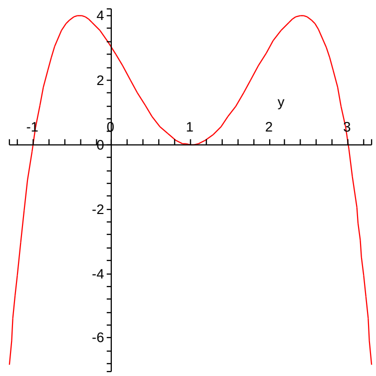
Math 2233, Exam I, Sept. 16, 2008 Name: ____________________

Score:

Part 1: Multiple choice. Each question is worth 4 points.

1. (e) Consider the autonomous differential equation $y' = f(y)$. The graph of $f(y)$ is given in below. Then, about the equilibrium solutions we know that:



(a) $y = -1$ and $y = 3$ are stable, $y = 1$ is semistable;

(b) $y = -1$ and $y = 3$ are unstable, $y = 1$ is semistable;

(c) $y = -1$ is stable, $y = 3$ is unstable, $y = 1$ is semistable;

(d) $y = 1$ is stable, $y = -1$ and $y = 3$ are semistable;

(e) $y = -1$ is unstable, $y = 3$ is stable, $y = 1$ is semistable.

2. (c) Which one of the following equations is a first order linear differential equation?

(a) $xy' + (\sin x)y = y^2$;

(b) $yy' + 6y = e^x$;

(c) $xy' + (\sin x)y = x^2$;

(d) $x^2y'' + xy' + 2y = e^x$;

(e) $xy + (\sin x)y = x^2$.

3. (e) If $y(t) = cos(3t)$ is a solution of $y'' - 9y = f(t)$, then $f(t) =$

(a) 0;

(b) $-8\cos(3t)$;

(c) $-10\cos(3t)$;

(d) $-12\cos(3t)$;

(e) $-18\cos(3t)$.

4. (b) Determine an interval in which the solution of the following initial value problem is certain to exist:

$$y' + \frac{t}{t+3}y = \frac{1}{t-2}, \qquad y(0) = 1$$

(a) $t < -3$;

(b) $-3 < t < 2$;

(c) $-2 < t < 3$;

(d) $2 < t < 3$;

(e) $t > 2$.

5. (d) The substitution $v = \frac{y}{x}$ transforms the equation $\frac{dy}{dx} = \sin(\frac{y}{x})$ into:

(a) $v' = sin(v)$;

(b) $v' = x\sin(v)$;

(c) $v' + v = \sin(v)$;

(d) $xv' + v = \sin(v)$;

(e) $v' + xv = \sin(v)$.

Part 2. Partial credit section. Show all your work neatly and concisely, and indicate your final answer clearly. If you simply write down the final answer without appropriate intermediate steps, you may not get full credit for that problem.

6. (8 points) Find the general solution of the equation:

$$\frac{dy}{dt} = \frac{2t}{y + t^2 y}.$$

(Hint: This equation is separable.)

Solution:

$$\begin{aligned} & \frac{dy}{dt} = \frac{2t}{y(1+t^2)} = \frac{1}{y}\frac{2t}{1+t^2} \\ \Rightarrow \quad & y\,dy = \frac{2t}{1+t^2}dt \\ \Rightarrow \quad & \int y\,dy = \int \frac{2t}{1+t^2}dt \\ \Rightarrow \quad & \frac{1}{2}y^2 = \ln|1+t^2| + C = \ln(1+t^2) + C \\ \Rightarrow \quad & y = \pm\sqrt{2\ln(1+t^2) + 2C} \end{aligned}$$

7. (8 points) Solve the initial value problem:

$$\begin{cases} y' + 2xy = x \\ y(0) = 2 \end{cases}$$

Solution 1: Use integrating factor

$$\begin{aligned}
& \mu(x) = e^{\int 2x\,dx} = e^{x^2} \\
\Rightarrow \quad & e^{x^2}(y' + 2xy) = xe^{x^2} \\
\Rightarrow \quad & (e^{x^2}y)' = xe^{x^2} \\
\Rightarrow \quad & e^{x^2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C \\
\Rightarrow \quad & y = \frac{1}{2} + Ce^{-x^2}
\end{aligned}$$

Plug in the initial condition $y(0) = 2$:

$$2 = \frac{1}{2} - C \qquad \Rightarrow \qquad C = -\frac{3}{2}$$

Solution to the initial value problem is

$$y = \frac{1}{2} + \frac{3}{2}e^{-x^2}$$

Solution 2: separable equation

$$\begin{aligned}
& y' + 2xy = x \\
\Rightarrow \quad & y' = x - 2xy = x(1 - 2y) \\
\Rightarrow \quad & \frac{1}{1-2y}dy = x\,dx \\
\Rightarrow \quad & \int \frac{1}{1-2y}dy = \int x\,dx \\
\Rightarrow \quad & -\frac{1}{2}\ln|1-2y| = \frac{1}{2}x^2 + C \\
\Rightarrow \quad & \ln|1-2y| = -x^2 - 2C \\
\Rightarrow \quad & |1-2y| = e^{-x^2-2C} = e^{-2C}e^{-x^2} \\
\Rightarrow \quad & 1 - 2y = ce^{-x^2} \qquad (\text{let } c = \pm e^{-2C}) \\
\Rightarrow \quad & y = \frac{1}{2} - \frac{c}{2}e^{-x^2}
\end{aligned}$$

Plug in the initial condition $y(0) = 2$:

$$2 = \frac{1}{2} - C \qquad \Rightarrow \qquad C = -\frac{3}{2}$$

Solution to the initial value problem is

$$y = \frac{1}{2} - (-\frac{3}{2})e^{-x^2} = \frac{1}{2} + \frac{3}{2}e^{-x^2}$$

8. (6 points) A tank with a capacity of 50 gallons originally contains 20 gallons of water with 10 pounds of salt in solution. Water containing 1 pound of salt per gallon is entering at a rate of 3 gal/min, and the well-stirred mixture is allowed to flow out of the tank at a rate of 2 gal/min. Let $Q(t)$ be the amount of salt in the tank at time t. Write an **initial value problem** for Q.

   (*You do NOT need to solve the initial value problem*)

   Solution: The initial value problem is

$$\begin{cases} \frac{dQ}{dt} = 3 \times 1 - 2 \times \frac{Q}{20+(3-2)t} = 3 - 2\frac{Q}{20+t} \\ Q(0) = 10 \end{cases}$$

9. (8 points) Solve the equation:

$$(\frac{y}{x} - 6x)dx + (\ln x - 2)dy = 0 \qquad \text{for } x > 0$$

Solution: We first check if this equation is exact. Notice that

$$M(x,y) = \frac{y}{x} - 6x, \qquad N(x,y) = \ln x - 2$$

Clearly

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}.$$

Hence this is an exact equation.

Next, we compute the potential $\phi(x,y)$ for this equation.

Step 1

$$\phi = \int M\,dx = \int (\frac{y}{x} - 6x)\,dx = y\ln|x| - 3x^2 + h(y)$$

Because $x > 0$, we can drop the $|\cdot|$ around x and then

$$\phi = y\ln x - 3x^2 + h(y)$$

Step 2 By using $\partial\phi/\partial y = N(x,y)$, we have

$$\frac{\partial\phi}{\partial y} = \ln|x| + h'(y) = N(x,y) = \ln x - 2.$$

Therefore we must have $h'(y) = -2$ and consequently $h(y) = -2y + c$.

Step 3 Combine step 1 & 2, we have

$$\phi = y\ln x - 3x^2 - 2y + c$$

An implicit solution to the equation is then given by $\phi = C$:

$$y\ln x - 3x^2 - 2y = C.$$